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## INVARIANT IDEMPOTENT MEASURES

The idempotent mathematics is a part of mathematics in which arithmetic operations in the reals are replaced by idempotent operations. In the idempotent mathematics, the notion of idempotent measure (Maslov measure) is a counterpart of the notion of probability measure. The idempotent measures found numerous applications in mathematics and related areas, in particular, the optimization theory, mathematical morphology, and game theory.

In this note we introduce the notion of invariant idempotent measure for an iterated function system in a complete metric space. This is an idempotent counterpart of the notion of invariant probability measure defined by Hutchinson. Remark that the notion of invariant idempotent measure was previously considered by the authors for the class of ultrametric spaces.

One of the main results is the existence and uniqueness theorem for the invariant idempotent measures in complete metric spaces. Unlikely to the corresponding Hutchinson's result for invariant probability measures, our proof does not rely on metrization of the space of idempotent measures.

An analogous result can be also proved for the so-called in-homogeneous idempotent measures in complete metric spaces.

Also, our considerations can be extended to the case of the max-min measures in complete metric spaces.

*Key words and phrases:* idempotent measure (Maslov measure), iterated function system, invariant measure.

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## INTRODUCTION

The idempotent mathematics is a part of mathematics in which arithmetic operations on the reals are replaced by idempotent operations (e.g., max, min; see [9]). According to an informal correspondence principle, every substantial notion of the (ordinary) mathematics has its counterpart in the idempotent mathematics. In this way we obtain the notion of idempotent measure, which is an idempotent analogue of that of probability measure. The idempotent measures found numerous applications, e.g. in the optimization theory, mathematical morphology, and game theory (see [2, 12–15]).

Different aspects of the theory of idempotent measures are considered in [1, 5, 8, 22]. In particular, the topology of spaces of the idempotent measures on some compact metric spaces is investigated in [5]. However, the theory of idempotent measures is considerably less developed than that of probability measures.

The mathematical foundations of the theory of deterministic fractals were created by Hutchinson [16]. In particular, he introduced the notions of invariant (self-similar) set and invariant measure for an iterated function system (IFS) of contractions on a complete metric space.

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The existence of invariant measures is proved in [16] by using the Banach contraction principle for suitable metrization of the set of probability measures on a metric space. The invariant measures impose an additional structure on the invariant set for the given IFS.

In [4], the authors considered a modification of the notions of invariant set and invariant probability measure, namely, the notions of in-homogeneous set and in-homogeneous probability measure (see also [17,18]). The inhomogeneous sets and measures are used, in particular, in image compression (see, e.g., [19]).

The aim of this note is to introduce the invariant idempotent measures for given IFS. In the case of idempotent measure, we use the weak\* convergence for proving the existence of invariant element. This approach seems to be fairly general and we anticipate new results in this direction (see the concluding remarks).

Note also that the invariant idempotent measures on ultrametric spaces are introduced and investigated in [11].

## 1 PRELIMINARIES

As usual,  $C(X)$  denotes the Banach space of continuous functions on a compact space  $X$ . We endow  $C(X)$  with the sup-norm. For any  $c \in \mathbb{R}$ , by  $c_X$  we denote the constant function on  $X$  taking the value  $c$ .

By  $\bar{A}$  we denote the closure of a set  $A$  in a topological space.

Let  $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ . We use the following operations  $\odot, \oplus$  of idempotent mathematics (see e.g., [9]):  $x \odot y = x + y$ , and  $x \oplus y = \max\{x, y\}$ ,  $x, y \in \mathbb{R}_{\max}$  (convention:  $(-\infty) \odot x = x \odot (-\infty) = -\infty$ ,  $(-\infty) \oplus x = x \oplus (-\infty) = x$ ). Also we consider the operations  $\odot: \mathbb{R} \times C(X) \rightarrow C(X)$ ,  $\lambda \odot \varphi = \lambda_X + \varphi$ , and  $\oplus: C(X) \times C(X) \rightarrow C(X)$ ,  $(\varphi \oplus \psi) = \max\{\varphi, \psi\}$ .

**Definition 1.1.** A functional  $\mu: C(X) \rightarrow \mathbb{R}$  is called an idempotent measure (a Maslov measure) if

1.  $\mu(c_X) = c$ ,
2.  $\mu(c \odot \varphi) = c \odot \mu(\varphi)$ , and
3.  $\mu(\varphi \oplus \psi) = \mu(\varphi) \oplus \mu(\psi)$

(see, e.g., [22] and references therein for the history and motivations of the notion of Maslov measure and Maslov integral).

By  $I(X)$  we denote the set of all idempotent measures on  $X$ .

Let  $\delta_x$  (or  $\delta(x)$ ) denote the Dirac measure concentrated at  $x \in X$ , i.e.,  $\delta_x(\varphi) = \varphi(x)$ ,  $\varphi \in C(X)$ . Clearly,  $\delta_x \in I(X)$ . A more complicated example of an idempotent measure is  $\mu = \bigoplus_{i=1}^n \alpha_i \odot \delta_{x_i}$ , where  $x_i \in X$  and  $\alpha_i \in \mathbb{R}_{\max}$ ,  $i = 1, \dots, n$ , and  $\bigoplus_{i=1}^n \alpha_i = 0$ .

We endow the set  $I(X)$  with the weak\* topology. In the case of compact metrizable space  $X$ , this topology is completely described by the convergent sequences:  $(\mu_i)_{i=1}^{\infty}$  converges to  $\mu$  if and only if  $\lim_{i \rightarrow \infty} \mu_i(\varphi) = \mu(\varphi)$ , for all  $\varphi \in C(X)$ .

Given a map  $f: X \rightarrow Y$  of compact Hausdorff spaces, the map  $I(f): I(X) \rightarrow I(Y)$  is defined by the formula  $I(f)(\mu)(\varphi) = \mu(\varphi f)$ , for every  $\mu \in I(X)$  and  $\varphi \in C(Y)$ . That  $I(f)$  is continuous and that  $I$  is a covariant functor acting in the category **Comp** of compact Hausdorff spaces and continuous maps was proved in [22].

If  $f: A \rightarrow X$  is an embedding of compact Hausdorff spaces, then so is the map  $I(f): I(A) \rightarrow I(X)$ . We identify  $I(A)$  and the subspace  $I(f)(I(A))$  via this embedding. The *support*  $\text{supp}(\mu)$  of an idempotent measure  $\mu \in I(X)$  is the minimal (with respect to inclusion) closed subset  $A$  in  $X$  such that  $\mu \in I(A)$ . According to [7] one can define the space  $I(X)$  also in non-compact case. If  $X$  is a Tychonov space, then let

$$I(X) = \{\mu \in I(\beta X) \mid \text{supp}(\mu) \subset X \subset \beta X\},$$

where  $\beta X$  stands for the Stone-Ćech compactification of  $X$ .

Recall that a map  $f: X \rightarrow Y$  of a metric space  $(X, d)$  into a metric space  $(Y, \rho)$  is called a contraction if there exists  $c \in (0, 1)$  such that  $\rho(f(x), f(y)) \leq cd(x, y)$ , for all  $x, y \in X$ .

By  $\exp X$  we denote the hyperspace of a topological space  $X$ , i.e., the set of all nonempty compact subsets of  $X$ . If  $(X, d)$  is a metric space, then  $\exp X$  is endowed with the Hausdorff metric  $d_H$ ,

$$d_H(A, B) = \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B), B \subset O_\varepsilon(A)\},$$

where  $O_r(C)$  stands for the  $r$ -neighborhood of a set  $C$  in  $X$ .

## 2 RESULT

Let  $X$  be a complete metric space and let  $f_1, \dots, f_n$  be an Iterated Function System (thereafter IFS) on  $X$ . We assume that all  $f_i$  are contractions. Let also  $\alpha_1, \dots, \alpha_n \in \mathbb{R}$  be such that  $\bigoplus_{i=1}^n \alpha_i = 0$ .

We denote by  $\Psi_0$  the identity map of  $\exp X$  and, for  $i > 0$ , define  $\Psi_i: \exp X \rightarrow \exp X$  inductively:  $\Psi_i(A) = \bigcup_{j=1}^n f_j(\Psi_{i-1}(A))$ .

Let  $\Phi_0: I(X) \rightarrow I(X)$  be the identity map. For  $i > 0$ , define  $\Phi_i: I(X) \rightarrow I(X)$  inductively:  $\Phi_i(\mu) = \bigoplus_{j=1}^n \alpha_j \odot I(f_j)(\Phi_{i-1}(\mu))$ . Thus,  $\Phi_i = \Phi_1 \Phi_1 \cdots \Phi_1$  ( $i$  times). It is easy to check that the maps  $\Phi_i$  are well-defined. In this case, we say that  $\mu \in I(X)$  is an *invariant idempotent measure* if  $\Phi_i(\mu) = \mu$  for every  $i = 0, 1, \dots$  (equivalently,  $\Phi_1(\mu) = \mu$ ).

Now, let  $\tau \in I(X)$  and let  $\alpha_1, \dots, \alpha_n, \alpha \in \mathbb{R}$  be such that  $(\bigoplus_{i=1}^n \alpha_i) \oplus \alpha = 0$ . Let  $\hat{\Phi}_0 = \Phi_0$  and define  $\hat{\Phi}_i: I(X) \rightarrow I(X)$  inductively:  $\hat{\Phi}_i(\mu) = \bigoplus_{j=1}^n \alpha_j \odot I(f_j)(\hat{\Phi}_{i-1}(\mu)) \oplus \alpha \odot \tau$ . Following the terminology of [17, 18] we say that  $\hat{\mu} \in I(X)$  is an *inhomogeneous invariant idempotent measure* if  $\hat{\mu} = \hat{\Phi}_1(\hat{\mu})$ .

**Theorem 1.** *There exists a unique invariant idempotent measure for the IFS  $f_1, \dots, f_n$  and  $\alpha_1, \dots, \alpha_n \in \mathbb{R}$  with  $\bigoplus_{i=1}^n \alpha_i = 0$ . This invariant measure is the limit of the sequence  $(\Phi_i(\mu))_{i=1}^\infty$ , for arbitrary  $\mu \in I(X)$ .*

*Proof.* Let  $\mu \in I(X)$ . We are going to prove that the sequence  $(\Phi_i(\mu)(\varphi))_{i=1}^\infty$  converges for arbitrary  $\varphi \in C(X)$ .

We first note that, without loss of generality, one may assume that  $X$  is compact. Indeed, for every  $i \geq 0$ , we see that

$$\text{supp}(\Phi_i(\mu)) \subset \Psi_i(\text{supp}(\mu)) \subset \overline{\bigcup_{j=0}^{\infty} \Psi_j(\text{supp}(\mu))}$$

and the latter set is compact by [16].

Let  $\varphi \in C(X)$  and let  $\varepsilon > 0$ . There exists  $\eta > 0$  such that, for every  $A \subset X$ ,  $\text{diam}(A) < \eta$  implies  $\text{diam}(\varphi(A)) < \varepsilon$ . There exists  $N \in \mathbb{N}$  such that for every  $k \geq N$ ,

$$\text{diam}(f_{i_1} \dots f_{i_k}(X)) < \eta,$$

for every  $i_1, \dots, i_k \in \{1, \dots, n\}$ . Then

$$\begin{aligned} \Phi_N(\mu)(\varphi) &= \bigoplus_{i_1, \dots, i_N} (\alpha_{i_1} \odot \dots \odot \alpha_{i_N}) \odot \mu(\varphi f_{i_1} \dots f_{i_N}) \\ &= (\alpha_{j_1} \odot \dots \odot \alpha_{j_N}) \odot \mu(\varphi f_{j_1} \dots f_{j_N}) \end{aligned}$$

for some  $j_1, \dots, j_N$ . By the choice of  $N$ ,

$$\varphi(x) - \varepsilon < \mu(\varphi f_{i_1} \dots f_{i_N}) < \varphi(x) + \varepsilon, \tag{1}$$

for every  $x \in f_{j_1} \dots f_{j_N}(X)$ . There is  $j$  such that  $\alpha_j = 0$ . Then, for every  $k \geq 1$ ,

$$\Phi_{N+k}(\mu)(\varphi) \geq (\alpha_{j_1} \odot \dots \odot \alpha_{j_N}) \odot \mu(\varphi f_{j_1} \dots f_{j_N} \underbrace{f_j \dots f_j}_k).$$

Then also  $\varphi(x) - \varepsilon < \mu(\varphi f_{j_1} \dots f_{j_N} \underbrace{f_j \dots f_j}_k) < \varphi(x) + \varepsilon$ , for every  $x \in f_{j_1} \dots f_{j_N} \underbrace{f_j \dots f_j}_k(X) \subset f_{j_1} \dots f_{j_N}(X)$ . We conclude that  $\Phi_{N+k}(\mu)(\varphi) \geq \Phi_N(\mu)(\varphi) - 2\varepsilon$  and, since the sequence  $(\Phi_i(\mu))$  is bounded, we conclude that there exists the limit of this sequence.

Now we are going to prove that the limit does not depend on the choice of  $\mu$ . Let also  $\nu \in I(X)$ . Again, without loss of generality, one may assume that  $X$  is compact. Indeed, one could let

$$X = \overline{\bigcup_{j=0}^{\infty} \Psi_j(\text{supp}(\mu) \cup \text{supp}(\nu))}.$$

Replacing  $\mu$  by  $\nu$  in (1) we obtain  $\Phi_{N+k}(\mu)(\varphi) \geq \Phi_N(\nu)(\varphi) - 2\varepsilon$  and therefore  $\lim_{k \rightarrow \infty} \Phi_k(\mu)(\varphi) = \lim_{k \rightarrow \infty} \Phi_{N+k}(\mu) \geq \Phi_N(\nu)(\varphi) - 2\varepsilon$ . From the latter inequality we obtain

$$\lim_{k \rightarrow \infty} \Phi_k(\mu)(\varphi) \geq \lim_{N \rightarrow \infty} \Phi_N(\nu)(\varphi) - 2\varepsilon$$

and, because of arbitrariness of  $\varepsilon > 0$ ,  $\lim_{k \rightarrow \infty} \Phi_k(\mu)(\varphi) \geq \lim_{N \rightarrow \infty} \Phi_N(\nu)(\varphi)$ .

Switching  $\mu$  and  $\nu$  we obtain the reverse inequality and therefore the equality.

Finally, the uniqueness of the invariant idempotent measure is an obvious consequence of the above established fact that the limit  $\lim_{i \rightarrow \infty} \Phi_i(\mu)$  does not depend on the choice of  $\mu$ . □

**Example 1.** Let  $X = [0, 1]$  and let  $f_1, f_2: X \rightarrow X$  be given by the formulas:  $f_1(t) = t/3$ ,  $f_2(t) = (t + 2)/3$ . The invariant set that corresponds to the IFS  $f_1, f_2$  is exactly the middle-third Cantor set.

Let  $\alpha_1 = 0$  and  $\alpha_2 = -1$ . Let  $\mu = \delta_0$ . Then, for every  $n \geq 1$ ,

$$\mu_n = 0 \odot \delta_0 \oplus \bigoplus_{1 \leq i_1 < \dots < i_k \leq n} (-k) \odot \delta \left( \sum_{j=1}^k \frac{2}{3^{i_j}} \right).$$

Then the invariant idempotent measure corresponding to  $\{f_1, f_2; \alpha_1, \alpha_2\}$  is

$$\lim_{n \rightarrow \infty} \mu_n = 0 \odot \delta_0 \oplus \bigoplus_{1 \leq i_1 < \dots < i_k} (-k) \odot \delta \left( \sum_{j=1}^k \frac{2}{3^{i_j}} \right).$$

One can similarly prove the following result.

**Theorem 2.** *There exists a unique inhomogeneous invariant idempotent measure for the IFS  $f_1, \dots, f_n$  and  $\alpha_1, \dots, \alpha_n, \alpha \in \mathbb{R}$  with  $(\bigoplus_{i=1}^n \alpha_i) \oplus \alpha = 0$ . This inhomogeneous invariant measure is the limit of the sequence  $(\hat{\Phi}_i(\mu))_{i=1}^\infty$ , for arbitrary  $\mu \in I(X)$ .*

### 3 MAX-MIN MEASURES

Let  $\bar{\mathbb{R}} = \mathbb{R}_{\max} \cup \{\infty\} = \mathbb{R} \cup \{-\infty, \infty\}$ . In the sequel,  $\otimes$  is used for min.

A functional  $\mu: C(X) \rightarrow \mathbb{R}$  is called a *max-min measure* if the following are satisfied:

1.  $\mu(c_x) = c$ ;
2.  $\mu(\varphi \oplus \psi) = \mu(\varphi) \oplus \mu(\psi)$ ;
3.  $\mu(c \otimes \varphi) = c \otimes \mu(\varphi)$

(see, e.g., [6] for details).

By  $J(X)$  we denote the set of all max-min measures on a compact Hausdorff space  $X$ . The set  $J(X)$  is endowed with the weak\*-topology. A base of this topology consists of the sets of the form

$$\{\mu \in J(X) \mid |\mu(\varphi_i) - \nu(\varphi_i)| < \varepsilon, i = 1, \dots, n\},$$

where  $\nu \in J(X)$ ,  $\varphi_i \in C(X)$ ,  $i = 1, \dots, n$ ,  $n \in \mathbb{N}$ . Every map  $f: X \rightarrow Y$  of compact Hausdorff spaces induces a map  $J(f): J(X) \rightarrow J(Y)$  defined as follows:  $J(f)(\mu)(\varphi) = \mu(\varphi f)$ . It is proved in [6] that  $J$  is a functor acting in the category **Comp**. Similarly as above, one can consider the spaces  $J(X)$  for Tychonov (in particular, metrizable) spaces  $X$ .

Let  $X$  be a complete metric space and let  $f_1, \dots, f_n$  be an IFS on  $X$ . We assume that all  $f_i$  are contractions. Let also  $\alpha_1, \dots, \alpha_n \in \bar{\mathbb{R}}$  be such that  $\bigoplus_{i=1}^n \alpha_i = \infty$ .

Let  $\Phi'_0: J(X) \rightarrow J(X)$  be the identity map. For  $i > 0$ , define  $\Phi'_i: J(X) \rightarrow J(X)$  inductively:  $\Phi'_i(\mu) = \bigoplus_{j=1}^n \alpha_j \otimes J(f_j)(\Phi'_{i-1}(\mu))$ . We say that  $\mu \in J(X)$  is an *invariant max-min measure* if  $\Phi'_i(\mu) = \mu$  for every  $i = 0, 1, \dots$  (equivalently,  $\Phi'_1(\mu) = \mu$ ).

The following can be proved similarly as Theorem 1.

**Theorem 3.** *There exists a unique invariant max-min measure for the IFS  $f_1, \dots, f_n$  and  $\alpha_1, \dots, \alpha_n \in \bar{\mathbb{R}}$  with  $\bigoplus_{i=1}^n \alpha_i = \infty$ . This invariant measure is the limit of the sequence  $(\Phi'_i(\mu))_{i=1}^\infty$ , for arbitrary  $\mu \in J(X)$ .*

The notion of inhomogeneous invariant max-plus measure can be defined similarly to that of inhomogeneous invariant idempotent measure. One can also formulate (and prove) a counterpart of Theorem 3 for the inhomogeneous invariant max-plus measures.

### 4 REMARKS AND OPEN QUESTIONS

Our construction is in a sense parallel to that of the invariant probability measure from [16]. The latter implicitly exploits the structure of monad for the probability measure functor  $P$  (more specifically, the so-called multiplication map  $P^2 \rightarrow P$ ) and, in our case, the definition of  $\Phi$  is based on the monad structure for the functor  $I$  (see [22]).

The proof of existence of the invariant probability measure implicitly uses the existence of a ‘nice’ functorial metrization of the spaces of probability measures of metric spaces. In particular, this metrization satisfies the property that the mentioned multiplication map  $P^2(X) \rightarrow P(X)$  is nonexpanding and it is well-known that the Kantorovich metrization is as required [16,21]. Note that a metrization of the spaces  $I(X)$  is constructed in [5]. However, it is not known whether the multiplication map  $I^2(X) \rightarrow I(X)$  is non-expanding, for a metric space  $X$ . Taras Banach informed the authors that one can construct a metrization of the spaces  $I(X)$  which allows for applying Banach’s contraction principle. As far as we know, his result is not published. Remark that the existence of invariant objects for IFSs in some general assumptions was considered in [3].

Some other generalizations can be made for the so called Lawson monads in the category **Comp** introduced by T. Radul [20].

Note that in [10] the first-named author considered the invariant inclusion hyperspaces for IFSs in complete metric spaces.

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Ідемпотентна математика є частиною математики, в якій арифметичні операції на множині дійсних чисел замінюються ідемпотентними операціями. У ідемпотентній математиці поняття ідемпотентної міри (міри Маслова) є відповідником поняття ймовірнісної міри. Ідемпотентні міри знайшли численні застосування в математиці та суміжних областях, зокрема, в теорії оптимізації, математичній морфології та теорії ігор.

У цій замітці ми запроваджуємо поняття інваріантної ідемпотентної міри для ітерованої системи функцій у повному метричному просторі. Це ідемпотентний аналог поняття інваріантної ймовірнісної міри, означеної Гатчінсоном. Зауважимо, що поняття інваріантної ідемпотентної міри раніше розглядалося авторами для класу ультраметричних просторів.

Одним з основних результатів є теорема існування та єдиності для інваріантних ідемпотентних мір у повних метричних просторах. На відміну від відповідного результату Гатчінсона для інваріантних ймовірнісних мір, наше доведення не опирається на метризацію простору ідемпотентних мір.

Аналогічний результат можна також довести для так званих неоднорідних ідемпотентних мір у повних метричних просторах.

Також наші міркування можна поширити на випадок max-min мір у повних метричних просторах.

*Ключові слова і фрази:* ідемпотентна міра (міра Маслова), система ітерованих відображень, інваріантна міра.