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PRYIMAK H.M.

ON APPROXIMATION OF HOMOMORPHISMS OF ALGEBRAS OF ENTIRE FUNCTIONS ON BANACH SPACES

It is known due to R. Aron, B. Cole and T. Gamelin that every complex homomorphism of the algebra of entire functions of bounded type on a Banach space X can be approximated in some sens by a net of point valued homomorphism. In this paper we consider the question about a generalization of this result for the case of homomorphisms to any commutative Banach algebra A. We obtained some positive results if A is the algebra of uniformly continuous analytic functions on the unit ball of X.

Key words and phrases: analytic functions on Banach space, homomorphisms of algebras of analytic functions, approximation property.

Vasyl Stefanyk Precarpathian National University, 57 Shevchenka str., 76018, Ivano-Frankivsk, Ukraine
E-mail: phm90@ukr.net

INTRODUCTION

Let X be a complex Banach space and $H_b(X)$ be the algebra of entire functions of bounded type on X, that is, $H_b(X)$ consists of all analytic functions on X which are bounded on all bounded sets. It is known that $H_b(X)$ is a Fréchet algebra with respect to the following family of norms

$$\|f\|_r = \sup_{\|x\|\le r} |f(x)|, \quad f \in H_b(x),$$

where r is taken over the set of positive rational numbers. We denote by M_b the spectrum of $H_b(X)$, that is, the set of all continuous complex valued homomorphisms of $H_b(X)$. M_b is a topological space endowed with the Gelfand topology which is the weakest topology such that all mappings $\widehat{f}(\varphi) := \varphi(f)$ are continuous. Typical examples of elements in M_b are point evaluation functionals δ_x, $x \in X$ which are defined by $\delta_x(f) = f(x)$, $f \in H_b(X)$.

In [1] it was proved that for every complex homomorphism $\varphi \in M_b$ there exists a net $(x_\alpha) \subset X$ such that $\varphi(P) = \lim\limits_{\alpha} P(x_\alpha)$ for every $P \in \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the algebra of all continuous polynomials on X. This property was used for investigations of spectra in [8, 9, 6, 3]. Our task is to generalize this formula in the case of homomorphisms from $H_b(X)$ to some commutative Banach algebra A.

Let A be a complex commutative Banach algebra and $A \otimes_\pi X$ be the complete projective tensor product of A and X. Every element of $A \otimes_\pi X$ can be represented by the form $\overline{a} = \sum\limits_k a_k \otimes x_k$, where $a_k \in A,\ x_k \in X$.

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For every $f \in H_b(X)$ let us define a function $\overline{f} : A \otimes_\pi X \to A$ so that for every $\overline{a} \in A \otimes_\pi X$, $\overline{f}(\overline{a})$ is the "value" of f at $\overline{a}$ in the means of functional calculus for analytic functions on a Banach spaces ([5]). Then the mapping $f \mapsto \overline{f}$ is a homomorphism between algebras $H_b(X)$ and $(H_b(A \otimes_\pi X), A)$. For every fixed $\overline{a}$ we define $\theta_{\overline{a}}(f) = \overline{f}(\overline{a})$ and θ is a homomorphism from $H_b(X)$ to A (see also [7, 10]).

The article is motivated by the following general question: *under which conditions for an arbitrary homomorphism Φ from $H_b(X)$ to A there exists a net $(\overline{a}_\alpha) \subset A \otimes_\pi X$ such that*

$$\Phi(P) = \lim_\alpha \theta_{\overline{a}}(P) = \lim_\alpha \overline{P}(\overline{a}_\alpha), \ \forall P \in \mathcal{P}(X)? \tag{1}$$

We obtain some positive answers under assumption that X has the approximation property for the case when $A = \mathcal{H}_{uc}^\infty(B)$, where $\mathcal{H}_{uc}^\infty(B)$ is the algebra of all uniformly continuous analytic complex functions on closed unit ball $B := \{x \in X : ||x|| \leq 1\}$ with norm

$$\|f\| = \sup_{\|x\| \leq 1} |f(x)|.$$

For more definitions and properties of polynomials and entire functions of bounded type on Banach spaces we refer the reader to [4].

1 MAIN RESULTS

We consider case when $A = \mathcal{H}_{uc}^\infty(B)$. Also, we suppose, first that Φ is the identity mapping, that is, $\Phi = I \colon H_b(X) \hookrightarrow \mathcal{H}_{uc}^\infty(B)$ and $I(f)$ is the restriction of f to B.

Our destination is to show that under some conditions there exists a net $(\overline{a}_\alpha) \in \mathcal{H}_{uc}^\infty(B) \otimes_\pi X$ such that

$$\Phi(f)(x) = \lim_\alpha \overline{f}(\overline{a}_\alpha) \quad \forall\, f \in H_b(X) \tag{2}$$

for $\Phi = I$ and for a more general case of Φ.

Example 1. *Let us consider $X = \mathbb{C}^n$ and $\Phi = I \colon H_b(\mathbb{C}^n) = H(\mathbb{C}^n) \hookrightarrow \mathcal{H}_{uc}^\infty(B) = \mathcal{A}(B)$.*

Every element $x \in \mathbb{C}^n$ can be represented as

$$x = \sum_{k=1}^n e_k^*(x) e_k,$$

where $\{e_k\}_{k=1}^n$ is a basis in $\mathbb{C}^n$ and $\{e_k^*\}_{k=1}^n$ is the dual basis of the coordinate functionals. Then $\overline{a} \in \mathcal{A}(B) \otimes_\pi \mathbb{C}^n$, $\overline{a} = \sum\limits_{k=1}^n e_k^* \otimes e_k$, that is

$$\overline{a}(x) = \sum_{k=1}^n e_k^*(x) e_k = x.$$

On the other hand, in the sense of functional calculus we have:

$$I(f)(x) = I(f(x)) = f(x) = f\left(\sum_{k=1}^n e_k^*(x) e_k\right) = \overline{f}(\overline{a})(x) = \overline{f}\left(\sum_{k=1}^n e_k^* \otimes e_k\right)(x).$$

Thus, for the fixed homomorphism $\Phi = I$ we found an element $\overline{a}$ and an arbitrary functions $f \in \mathcal{A}(B)$ and we have equality:

$$I(f(x)) = \overline{f}(\overline{a})(x) = \theta_{\overline{a}}(f)(x).$$

Note that in this case we need just a single A-evaluation functional $\theta_{\overline{a}}$.

Let Φ be an arbitrary homomorphism from $H(\mathbb{C}^n)$ to $\mathcal{A}(B)$ such that there is an analytic automorphism $F : B \to B$ such that $\Phi = C_F \circ I$, where C_F is the composition operator, $C_F(f)(x) = f(F(x)),\, f \in \mathcal{A}(B),\ x \in B$. We set

$$\overline{a} = \sum_{k=1}^{n} (e_k^* \circ F) \otimes e_k \ \in \mathcal{A}(B) \otimes \mathbb{C}^n.$$

Then $\Phi(f)(x) = \overline{f}(\overline{a})(x)$.

This example can be generalized to the case when X has a Schauder basis. Recall that the sequence $\{e_n\}_{n=1}^{\infty}$ in a Banach space is called a Schauder basis of X if for any $x \in X$ there exists a unique sequence of scalars $\{x_n\}_{n=1}^{\infty}$ such that

$$x = \sum_{n=1}^{\infty} x_n e_n,$$

and the series converges by the norm of X, that is,

$$\lim_{n\to\infty} \left\| x - \sum_{k=1}^{n} x_k e_k \right\| = 0.$$

We denote by e_n^* the coordinate functionals, $e_n^*(x) = x_n$.

Proposition 1. *Let X be a Banach space with a Schauder basis, $A = \mathcal{H}_{uc}^{\infty}(B)$, $\Phi = I \colon H_b(X) \to \mathcal{H}_{uc}^{\infty}(B)$. Then* (2) *holds for a sequence $\overline{a}_m \in \mathcal{H}_{uc}^{\infty}(B) \otimes_{\pi} X$.*

Proof. Let $\{e_k\}_{k=1}^{\infty}$ is a Schauder basis in X. Then every element $x \in X$ can be represented as $x = \sum\limits_{k=1}^{\infty} e_k^*(x) e_k$. Consider

$$\overline{a}_m = \sum_{k=1}^{m} e_k^* \otimes e_k = \sum_{k=1}^{m} e_k^* e_k.$$

In the sense of functional calculus we have:

$$\overline{f}(\overline{a}_m)(x) = \overline{f}\left(\sum_{k=1}^{m} e_k^* \otimes e_k\right)(x) = f\left(\sum_{k=1}^{m} e_k^*(x) e_k\right) = f\left(\sum_{k=1}^{m} x_k e_k\right).$$

Since $\{e_k\}_{k=1}^{\infty}$ is a Schauder basis, $\sum\limits_{k=1}^{m} x_k(e_k) \to x$ as $m \to \infty$. This means that

$$I(f)(x) = \lim_{m\to\infty} \overline{f}(\overline{a}_m)(x) = \lim_{m\to\infty} \theta_{\overline{a}_m}(f).$$

□

In the general case we consider the space with the approximation property.

Definition 1. *A Banach space X is said to have the approximation property in the sense of Grothendieck if for every compact set K in X and every $\varepsilon > 0$ there is an operator $T : X \to X$ of finite rank such that $\|Tx - x\| \le \varepsilon$ for every $x \in K$.*

Theorem 1. *Let X be a Banach space with the approximation property. Then for $\Phi = I$ equality* (2) *holds.*

Proof. Let $\mathfrak{A}$ be the following set of indexes: if $\alpha \in \mathfrak{A}$, then $\alpha = (K, \varepsilon, n)$, where K is a compact set in X, $\varepsilon > 0$ and $n \in \mathbb{N}$. We introduce a partial order on $\mathfrak{A}$ by the following way: $\alpha_1 \le \alpha_2$ if and only if $K_1 \subset K_2$, $\varepsilon_1 \le \varepsilon_2$ and $n_1 \le n_2$. So $\mathfrak{A}$ is a directed set. Since X has the approximation property, for every $\alpha = (K_\alpha, \varepsilon_\alpha, n_\alpha) \in \mathfrak{A}$ there is an operator T_α with the rank n_α such that for every $x \in K_\alpha$, $\|T_\alpha x - x\| \le \varepsilon_\alpha$. $(T_\alpha)_\alpha$ is a net and $x = \lim_\alpha T_\alpha x$ for every $x \in X$.

Let $\{\gamma_{k,\alpha}\}_{k=1}^{n_\alpha}$ be a basis in the range of T_α in X and $\{\gamma^*_{k,\alpha}\}_{k=1}^{n_\alpha} \in X'$ be linear functionals which are bi-orthogonal to $\{\gamma_{k,\alpha}\}_{k=1}^{n_\alpha}$. So $T_\alpha(x) = \sum_{k=1}^{n_\alpha} \gamma^*_{k,\alpha}(x)\gamma_{k,\alpha}$. Thus we can set

$$\overline{a}_\alpha = \sum_{k=1}^{n_\alpha} \gamma^*_{k,\alpha} \otimes \gamma_{k,\alpha},$$

Hence, for every $f \in H_b(X)$

$$I(f)) = \lim_\alpha \overline{f}(\overline{a}_\alpha) \in \mathcal{H}^\infty_{uc}(B)$$

and so equality (2) holds. □

It seems to be that the approximation property is to strong condition for our purpose. Let us consider the weak H_b topology on X as the the restriction of the Gelfand topology on X, that is, the weakest topology on X such that all $f \in H_b(X)$ are continuous.

Definition 2. *We say that X has the H_b-approximation property if for every compact set K in the weak $H_b(X)$ topology and every $\varepsilon > 0$ there exists a finite rank operator T such that*

$$|f(T(x) - x)| < \varepsilon$$

for every polynomial $f \in H_b(X)$ and every $x \in K$.

Doing the same work like in Theorem 1 we can prove the following theorem.

Theorem 2. *If X has the H_b-approximation property, then* (2) *holds.*

It is easy to see that every Banach space X with the approximation property has the H_b-approximation property but we do not know about the inverse implication. Also, we do not know any examples for which the property (2) is not true.

Let us consider (1) for more general case.

Theorem 3. *Let Φ be a homomorphism from $H_b(X)$ to $\mathcal{H}^\infty_{uc}(B)$ such that there is an analytic automorphisms $F : B \to B$ with $\Phi = C_F \circ I$, where C_F is the composition operator with F. Then* (2) *holds.*

Proof. Let

$$\sum_{k=1}^{n_\alpha} \gamma^*_{k,\alpha} \otimes \gamma_{k,\alpha}$$

be the net which approximate the identity map I as in the proof of Theorem 1. It enough to put

$$\overline{a}_\alpha = \sum_{k=1}^{n_\alpha} (\gamma^*_{k,\alpha} \circ F) \otimes \gamma_{k,\alpha}.$$

□

Note that in the general case, not every homomorphism Φ can be represented as $\Phi = C_F \circ I$. In [2] some related problems to the question about representation of homomorphisms by compositions operator were considered.

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Приймак Г.М. *Про наближення гомоморфізмів алгебри цілих функцій на банахових просторах* // Карпатські матем. публ. — 2019. — Т.11, №1. — C. 158–162.

Завдяки Р. Арону, Б. Коулу і Т. Гамеліну відомо, що кожен комплексний гомоморфізм алгебри цілих функцій обмеженого типу на банаховому просторі X можна наблизити в деякому сенсі за допомогою напрямлености поточкових гомоморфізмів. У даній роботі ми розглянемо питання про узагальнення цього результату для випадку гомоморфізмів зі значеннями у довільній комутативній банаховій алгебрі. Ми отримали деякі позитивні результати у випадку коли A — алгебра рівномірно неперервних аналітичних функцій на одиничній кулі простору X.

Ключові слова і фрази: аналітичні функції на банаховому просторі, гомоморфізми алгебри аналітичних функцій, властивість апроксимації.