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ON THE SOLUTIONS OF A CLASS OF NONLINEAR INTEGRAL EQUATIONS IN CONE b-METRIC SPACES OVER BANACH ALGEBRAS

In this paper, we study the existence of the solutions of a class of functional integral equations by using some fixed point results in cone b-metric spaces over Banach algebras. In order to obtain these results we introduced and proved some properties of generalized weak φ-contractions, in which the φ are nonlinear weak comparison functions. The obtained results are generalizations of results of Van Dung N., Le Hang V. T., Huang H., Radenovic S. and Deng G. Also, some suitable examples are given to illustrate obtained results.

Key words and phrases: cone b-metric space over Banach algebra, φ-contraction, **c**-sequence, fixed point, integral equation.

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1 INTRODUCTION AND PRELIMINARIES

In 2007, Huang and Zhang [5] introduced the concept of a cone metric space and generalized Banach fixed point theorem in such spaces. Afterwards, several authors published many papers on this topic. Aydi *et al.* [1, 2] proved some coupled coincidence point results on generalized distance in ordered cone metric spaces. Dordević *et al.* [4] considered fixed point and common fixed point results for maps in *tvs*-cone metric spaces under contractive conditions expressed in the terms of c-distance. By using an old Krein's result and a result concerning symmetric spaces, Jankovic *et al.* [10] showed in a very short way that fixed point results in cone metric spaces obtained recently, in which the assumption that the underlying cone is normal and solid is present, can be reduced to the corresponding results in metric spaces.

In 2013, Lia and Xu [12] introduced the notion of cone metric spaces over Banach algebras and defined a generalized Lipschitz contraction with vector contractive coefficient instead of usual real constant. The authors proved the existence of fixed points with the assumption that the underlying cone is normal. Furthermore, they explained by an example that the fixed point theorems in cone metric spaces over Banach algebra are not equivalent to those in metric spaces, and so, such generalizations are the genuine ones. Latter, Xu and Radenović [16] showed that the normality of the cone can be removed from the results of Liu and Xu [12]. In 2015, Huang and Radenović [6] introduced the notion of cone b-metric spaces over Banach algebra and presented some common fixed point theorems in such spaces. Subsequently, Huang and Radenović [7] considered the Banach type version of a fixed point result with the generalized Lipschitz constant k satisfying $\rho(k) \in [0, \frac{1}{s})$ where $\rho(k)$ is the spectral radius of k. In 2017, Huang *et al.* [8] generalized a famous result for Banach-type contractive map from $\rho(k) \in [0, \frac{1}{s})$

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to $\rho(k) \in [0,1)$ in cone b-metric spaces over Banach algebra with coefficient $s \geq 1$. Very recent, by using a nontrivial proof method Li and Huang [11] proved some fixed point results for weak φ-contractions in cone metric spaces over Banach algebras and applied to investigate the existence and uniqueness of a solution to two classes of equations. However, in the construction of such applications, the functions φ considered in φ-contractions are simple linear functions, for example see [6, Theorem 3.1] and [11, Theorem 3.2].

In this paper, we study the existence of the solutions of a class of functional integral equations by using some fixed point results in cone b-metric spaces over Banach algebras. In order to obtain these results we introduced and proved some properties of generalized weak φ-contractions, in which the φ are nonlinear weak comparison functions, and we also illustrated obtained results by suitable examples.

Now we recall definitions and properties which will be useful in what follows.

Definition 1 ([14, p. 245]). *Let $(\mathcal{A}, \|.\|)$ be a Banach space over the real field $\mathbb{R}$ in which a multiplication is defined that for all $x, y, z \in A$ and for all $\alpha \in \mathbb{R}$ satisfies*

1) $(xy)z = x(yz)$,

2) $x(y+z) = xy + xz$ *and* $(x+y)z = xz + yz$,

3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$,

4) $\|xy\| \leq \|x\|\|y\|$,

5) *there is a unit element* e *with* $\|e\| = 1$ *such that* $xe = ex = x$.

Then $\mathcal{A}$ is called a Banach algebra.

Definition 2 ([7, p. 567]). *Let $\mathcal{A}$ be a Banach algebra with a unit e and a zero element θ. A nonempty closed subset P of $\mathcal{A}$ is called a cone in $\mathcal{A}$ if*

1) $\{\theta, e\} \subset P$,

2) $\alpha P + \beta P \subset P$, *for all* $\alpha, \beta \in \mathbb{R}_+$,

3) $P^2 = PP \subset P$,

4) $P \cap (-P) = \{\theta\}$.

Definition 3 ([7, p. 567]). *Let $\mathcal{A}$ be a Banach algebra and P is a cone in $\mathcal{A}$. We say that*

1) *P is a solid cone if* $\operatorname{int} \mathrm{P} \neq \varnothing$, *where* $\operatorname{int} \mathrm{P}$ *denotes the interior of P;*

2) *P is a normal cone if there is a number $M > 0$ such that for all $x, y \in \mathcal{A}$*

$$\theta \preceq x \preceq y \text{ implies } \|x\| \leq M\|y\|,$$

where $\|.\|$ is the norm in $\mathcal{A}$. The least positive value of M satisfying the above inequality is called the normal constant.

Note that, for any normal cone P we have $M \geq 1$ (see [13]).

For a given cone $P \subset \mathcal{A}$, we can define a partial ordering " $\preceq$ " with respect to P by $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$, if $x \preceq y$ and $x \neq y$, and denote $x \ll y$ if and only if $y - x \in \text{int}P$.

In the sequel, unless otherwise specified, we always suppose that $\mathcal{A}$ is a Banach algebra, P is a solid cone in $\mathcal{A}$, and $\preceq$, $\ll$ are the above partial orderings with respect to P.

Definition 4 ([5]). *Let X be a nonempty set, $\mathcal{A}$ be a Banach algebra and $d : X \times X \to \mathcal{A}$ be a map such that for all $x, y, z \in X$*

1) $\theta \preceq d(x,y)$, and $d(x,y) = \theta$ if and only if $x = y$,

2) $d(x,y) = d(y,x)$,

3) $d(x,z) \preceq d(x,y) + d(y,z)$.

Then d is called a cone metric on X and $(X, \mathcal{A}, d)$ is called a cone metric space over Banach algebra.

Definition 5 ([5]). *Let $(X, \mathcal{A}, d)$ be a cone metric space over Banach algebra, $\{x_n\}$ be a sequence in X and $x \in X$. Then*

1) $\{x_n\}$ converges to $x \in X$ if for each $c \in intP$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n \geq N$. Then, we write $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$;

2) $\{x_n\}$ is a Cauchy sequence if for each $c \in intP$ there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$;

3) $(X, \mathcal{A}, d)$ is called complete if each Cauchy sequence is convergent in X.

Definition 6 ([7]). *Let X be a nonempty set, $s \geq 1$ be a constant, $\mathcal{A}$ be a Banach algebra and $d : X \times X \to \mathcal{A}$ be a map such that for all $x, y, z \in X$*

1) $0 \preceq d(x,y)$, and $d(x,y) = 0$ if only if $x = y$,

2) $d(x,y) = d(y,x)$,

3) $d(x,z) \preceq s[d(x,y) + d(y,z)]$.

Then d is called a cone b-metric on X and $(X, \mathcal{A}, d, s)$ is called a cone b-metric space over Banach algebra with the coefficient s.

Remark 1 ([7]). *A cone metric space over Banach algebra must be a cone b-metric space over Banach algebra. Conversely, it is not true. As a result, the notion of cone b-metric space over Banach algebra greatly generalizes the notion of cone metric space over Banach algebra.*

The following example shows that there exists a cone b-metric spaces over Banach algebras which are not cone metric spaces over Banach algebras.

Example 1 ([7]). *Let $\mathcal{A} = C[0,1]$ be the usual Banach space with the supremum norm. Define multiplication in the usual way: $(xy)(t) = x(t)y(t)$, $t \in [0,1]$. Then $\mathcal{A}$ is a Banach algebra with a unit $e = 1$. Put $P = \{x \in \mathcal{A} : x(t) \geq 0, t \in [0,1]\}$ and $X = \mathbb{R}$. Define a map $d : X \times X \to \mathcal{A}$ by $d(x,y)(t) = |x-y|^p e^t$ for all $x, y \in X$, where $p > 1$ is a constant. This makes $(X, \mathcal{A}, d, s)$ into a cone b-metric space over Banach algebra with the coefficient $s = 2^{p-1}$, but it is not a cone metric space over Banach algebra.*

Similar to Definition 5, we repeat the notions of convergent sequence, Cauchy sequence and complete space in cone b-metric space over Banach algebra.

Definition 7 ([7]). *Let $(X, \mathcal{A}, d, s)$ be a cone b-metric space over Banach algebra and $\{x_n\}$ be a sequence in X. We say that*

1) $\{x_n\}$ converges *to $x \in X$ if for each $c \in intP$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n \geq N$. Then, we write $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$;*

2) *$\{x_n\}$ is a Cauchy sequence if for each $c \in intP$ there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$;*

3) *$(X, \mathcal{A}, d)$ is a complete cone b-metric space if each Cauchy sequence in X is convergent.*

Definition 8 ([4, Sect. 3.1]). *A sequence $\{u_n\} \subset P$ is called a* **c**-*sequence if for each $c \in \operatorname{int} P$, there exists $N \in \mathbb{N}$ such that $u_n \ll c$ for all $n > N$.*

Lemma 1 ([7]). *Let P be a solid cone in a Banach algebra $\mathcal{A}$, $\{u_n\}$ and $\{v_n\}$ be two* **c**-*sequences in P. If $\alpha, \beta \in P$ are two arbitrarily given vectors, then $\{\alpha u_n + \beta v_n\}$ is a* **c**-*sequence.*

Lemma 2 ([14]). *Let $\mathcal{A}$ be a Banach algebra. Then the spectral radius of $k \in \mathcal{A}$ equals to $\rho(k) = \lim\limits_{n\to\infty} \|k^n\|^{\frac{1}{n}} = \inf\limits_{n\geq 1} \|k^n\|^{\frac{1}{n}}$.*

Lemma 3 ([6]). *Let $\mathcal{A}$ be a Banach algebra. Let $k \in \mathcal{A}$ and $\rho(k) < 1$. Then $\{k^n\}$ is a* **c**-*sequence.*

Lemma 4 ([9]). *Let $\mathcal{A}$ be a Banach algebra and $u, v, w \in \mathcal{A}$. Then*

(1) if $u \preceq v$ and $v \ll w$, then $u \ll w$;

(2) If $u \ll v$ and $v \ll w$, then $u \ll w$;

(3) If $\theta \preceq u \ll c$ for each $c \in intP$, then $u = \theta$;

(4) $\alpha intP \subseteq intP$ for all $\alpha > 0$;

(5) If $c \in intP$, $\theta \preceq a_n$ and $\lim\limits_{n\to\infty} a_n = \theta$ then there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have $a_n \ll c$.

Definition 9 ([9]). *Let $(X, \mathcal{A}, d, s)$ be a cone b-metric space over Banach algebra and $B \subseteq X$. An element $b \in B$ is called an interior point of B whenever there is $\theta \ll p$ such that $B_0(b,p) \subseteq B$, where $B_0(b,p) = \{y \in X : d(y,b) \ll p\}$.*

Definition 10 ([15, p. 246]). *A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is called a comparison function if γ is increasing and $\lim\limits_{n\to\infty} \gamma^n(u) = 0$ for all $u \in \mathbb{R}_+$.*

The following some notions and property are well known in [11].

Definition 11 ([11]). *Let $\mathcal{A}$ be a Banach algebra and P be a cone in $\mathcal{A}$. A map $\varphi : P \to P$ is called a weak comparison if the following conditions hold*

(1) φ is nondecreasing with respect to $\preceq$, that is, for all $t_1, t_2 \in P$ and $t_1 \preceq t_2$, implies that $\varphi(t_1) \preceq \varphi(t_2)$;

(2) $\{\varphi^n(t)\}$ is a **c***-sequence in P for all $t \in P$;*

(3) if $\{u_n\}$ is a **c***-sequence in P, then $\{\varphi(u_n)\}$ is also a* **c***-sequence in P.*

Definition 12 ([11]). *Let $(X, \mathcal{A}, d)$ be a cone metric space over Banach algebra and P be a cone in $\mathcal{A}$. Let $\varphi : P \to P$ be a weak comparison. Then a map $f : X \to X$ is called a weak φ-contraction if for all $x, y \in X$,*

$$d\big(f(x), f(y)\big) \preceq \varphi\big(d(x,y)\big).$$

Theorem 1 ([11]). *Let $(X, \mathcal{A}, d)$ be a complete cone metric space over Banach algebra and $f : X \to X$ be a weak φ-contraction. Then f has a unique fixed point $u \in X$ and $\lim_{n\to\infty} f^n(x) = u$ for each $x \in X$.*

2 FIXED POINT RESULTS IN CONE b-METRIC SPACES OVER BANACH ALGEBRAS

First we extend the notion of weak φ-contraction in metric spaces to the setting of cone b-metric spaces over Banach algebra as follows.

Definition 13. *Let $(X, \mathcal{A}, d, s)$ be a cone b-metric space over Banach algebra and P be a cone in $\mathcal{A}$. Let $\varphi : P \to P$ be a weak comparison. Then a map $f : X \to X$ is called a generalized weak φ-contraction if for all $x, y \in X$,*

$$d\big(f(x), f(y)\big) \preceq \varphi\big(d(x,y)\big).$$

Lemma 5. *Let $(X, \mathcal{A}, d, s)$ be a cone b-metric space over Banach algebra, P be a cone in $\mathcal{A}$, and $f : X \to X$ be a generalized weak φ-contraction. Then,*

(1) for all $t_1, t_2 \in P$ with $t_1 \preceq t_2$ and all $n \in \mathbb{N}$, we have $\varphi^n(t_1) \preceq \varphi^n(t_2)$;

(2) for all $x, y \in X$ and all $n \in \mathbb{N}$, we have

$$d\big(f^n(x), f^n(y)\big) \preceq \varphi^n\big(d(x,y)\big).$$

Proof. (1). For any $t_1, t_2 \in P$ with $t_1 \preceq t_2$, since φ is a weak comparison, we have

$$\varphi(t_1) \preceq \varphi(t_2).$$

Then, we get

$$\varphi^2(t_1) = \varphi\big(\varphi(t_1)\big) \preceq \varphi\big(\varphi(t_2)\big) = \varphi^2(t_2).$$

Continuing the above process, we obtain that for all n,

$$\varphi^n(t_1) \preceq \varphi^n(t_2).$$

(2). For any $x, y \in X$, since f is a generalized weak φ-contraction, we have

$$d\big(f(x), f(y)\big) \preceq \varphi\big(d(x,y)\big).$$

Note that φ is a weak comparison, so we have

$$\varphi\Big(d\big(f(x), f(y)\big)\Big) \preceq \varphi\Big(\varphi\big(d(x,y)\big)\Big) = \varphi^2\big(d(x,y)\big). \tag{1}$$

Using f being a generalized weak φ-contraction again, we get

$$d\big(f^2(x), f^2(y)\big) \preceq \varphi\Big(d\big(f(x), f(y)\big)\Big). \tag{2}$$

From (1) and (2) we have

$$d\big(f^2(x), f^2(y)\big) \preceq \varphi^2\big(d(x,y)\big).$$

Continuing this process we obtain that for all n,

$$d\big(f^n(x), f^n(y)\big) \preceq \varphi^n\big(d(x,y)\big).$$

□

Now, we establish some results for generalized weak φ-contraction maps in complete cone b-metric space over Banach algebra.

Lemma 6. *Let $(X, \mathcal{A}, d, s)$ be a complete cone b-metric space over Banach algebra and $f : X \to X$ be a generalized weak φ-contraction. Then f has a unique fixed point $u \in X$ and for each $x \in X$, $\lim_{n\to\infty} f^n(x) = u$.*

Proof. Let any $x \in X$ and put $x_0 = x$, $x_n = f^n(x)$ for all $n \geq 1$.

Then, by Definition 13, for each $c \in \text{int}P$, exists $n_0 \in \mathbb{N}$ such that $\varphi^{n_0}(c) \ll s^{-1}c$. Using Lemma 5.(2), for every $n \in \mathbb{N}$ we have

$$d(x_n, x_{n+n_0}) \preceq \varphi^n\Big(d\big(x_0, x_{n_0}\big)\Big). \tag{3}$$

Since $\{\varphi^n\big(d(x_0, x_{n_0})\big)\}$ is a **c**-sequence then by (3) and Lemma 4.(1), we have $\{d(x_n, x_{n+n_0})\}$ is also a **c**-sequence. Hence, exists $N_1 \in \mathbb{N}$ such that

$$d(x_n, x_{n+n_0}) \ll s^{-1}c - \varphi^{n_0}(c) \text{ for all } n \geq N_1.$$

Put

$$B(x_n, c) = \{y \in X : d(x_n, y) \ll c\} \text{ for all } n \geq N_1 - 1. \tag{4}$$

For each $n \geq N_1 - 1$, choosing $y \in B(x_n, c)$, by (3) and (4) we have

$$\begin{aligned} d(x_n, f^{n_0}y) &\preceq s[d(x_n, x_{n+n_0}) + d(x_{n+n_0}, f^{n_0}y)] \\ &\preceq s[s^{-1}c - \varphi^{n_0}(c) + \varphi^{n_0}(d(x_n, y))] \\ &\ll c - s\varphi^{n_0}(c) + s\varphi^{n_0}(c) \\ &= c. \end{aligned}$$

This implies that $B(x_n, c)$ is f^{n_0}-invariant. Hence, for each $k \in \mathbb{N}$, we have

$$d(x_n, x_{n+kn_0}) = d(x_n, f^{n_0}x_n) \ll c, \text{ for all } n \geq N_1 - 1. \tag{5}$$

Using Lemma 5.(2), for every $n \in \mathbb{N}$ we get

$$\begin{aligned} &sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^{n_0} d(x_{n+n_0-1}, x_{n+n_0}) \\ &\qquad \preceq s\varphi^n\big(d(x_0, x_1)\big) + s^2\varphi^n\big(d(x_1, x_2)\big) + \cdots + s^{n_0}\varphi^n\big(d(x_{n_0-1}, x_{n_0})\big). \end{aligned} \tag{6}$$

For each $i = 0, 1, 2, ..., n_0$, we have $\{\varphi^n\big(d(x_i, x_{i+1})\big)\}$ is a **c**-sequences then by Lemma 1, $\{s\varphi^n\big(d(x_0, x_1)\big) + s^2\varphi^n\big(d(x_1, x_2)\big) + \cdots + s^{n_0}\varphi^n\big(d(x_{n_0-1}, x_{n_0})\big)\}$ is a **c**-sequence. Hence, by (6) and Lemma 4.(1), we have

$$\{sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^{n_0} d(x_{n+n_0-1}, x_{n+n_0})\}$$

is also a **c**-sequence. So, for any $c \in \text{int}P$, exists $N_2 \in \mathbb{N}$ such that

$$sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^{n_0} d(x_{n+n_0-1}, x_{n+n_0}) \ll c \tag{7}$$

for all $n \geq N_2$.

Denote $N = \max\{N_1, N_2\}$, for all $m,\ n > N$ we put

$$k_m = \Big[\frac{m-N}{n_0}\Big], \quad k_n = \Big[\frac{n-N}{n_0}\Big],$$

where $[\,.\,]$ stands for the integer part. Because

$$N \leq m - k_m n_0 < N + n_0, \quad N \leq n - k_n n_0 < N + n_0, \tag{8}$$

from (8) we find that

$$\big|(n - k_n n_0) - (m - k_m n_0)\big| < n_0.$$

Hence, from (7) we have

$$d(x_{n-k_n n_0}, x_{m-k_m n_0}) \preceq sd(x_{n-k_n n_0}, x_{n-k_n n_0+1}) + \cdots + s^{n_0} d(x_{n-k_n n_0+n_0-1}, x_{n-k_n n_0+n_0}) \ll c. \tag{9}$$

Hence, from (5) and (9) we find that

$$\begin{aligned} d(x_n, x_m) &\preceq sd(x_n, x_{n-k_n n_0}) + s^2 d(x_{n-k_n n_0}, x_{m-k_m n_0}) + s^2 d(x_{m-k_m n_0}, x_m) \\ &\ll (s + s^2 + s^2)c. \end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in $(X, \mathcal{A}, d, s)$. Since $(X, \mathcal{A}, d, s)$ is complete there exists $u \in X$ such that $\lim\limits_{n\to\infty} x_n = u$.

Next, we prove that u is the fixed point of f. Indeed, we have

$$\begin{aligned} d(fu, u) &\preceq sd(fu, x_n) + sd(x_n, u) \\ &= sd(fu, fx_{n-1}) + sd(x_n, u) \\ &\preceq s\varphi\big(d(u, x_{n-1})\big) + sd(x_n, u). \end{aligned} \tag{10}$$

Since $\{d(x_n, u)\}$ is a **c**-sequence and φ is weak comparison, then $\{\varphi\big(d(u, x_{n-1})\big)\}$ is also a **c**-sequence. Hence, by Lemma 1 we have $\{s\varphi\big(d(u, x_{n-1})\big) + sd(x_n, u)\}$ is a **c**-sequence. By (10), Lemma 4.(3) and $\{s\varphi\big(d(u, x_{n-1})\big) + sd(x_n, u)\}$ is a **c**-sequence, we find that $d(fu, u) = \theta$. This implies that u is a fixed point of f.

Finally, we prove that the fixed point is unique. Assume that v is another fixed point of f. Then we have

$$\theta \preceq d(u, v) = d\big(f^n(u), f^n(v)\big) \preceq \varphi^n\big(d(u, v)\big) \quad \text{for all } n \geq 1. \tag{11}$$

Since $\{\varphi^n\big(d(u, v)\big)\}$ is a **c**-sequence then by (11) and Lemma 4.(3), we have $d(u, v) = \theta$. This implies that $u = v$.

So, f have unique fixed point $u \in X$ and for each $x \in X$, $\lim\limits_{n\to\infty} f^n(x) = u$. □

In Lemma 6, if we choose $\varphi : P \to P$ by $\varphi(t) = kt$, for all $t \in \mathcal{A}$ and $k \in P$ such that $\rho(k) < 1$, then we obtain the following.

Corollary 1 ([8]). *Let $(X, \mathcal{A}, d, s)$ be a complete cone b-metric space over Banach algebra and $f : X \to X$ be a map such that for all $x, y \in X$,*

$$d(f(x), f(y)) \preceq kd(x, y), \tag{12}$$

where $k \in P$ is a generalized Lipschitz constant with $\rho(k) < 1$. Then, f has a unique fixed point $u \in X$ and for each $x \in X$, $\lim\limits_{n\to\infty} f^n(x) = u$.

By choosing $\mathcal{A} = \mathbb{R}$ and $P = \mathbb{R}_+$ in Lemma 6, then we obtain the following.

Corollary 2 ([3]). *Let (X, d, s) be a complete b-metric space and $f : X \to X$ be a map such that for all $x, y \in X$,*

$$d(f(x), f(y)) \leq \varphi(d(x, y)),$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function. Then, f has a unique fixed point $u \in X$ and for each $x \in X$, $\lim\limits_{n\to\infty} f^n(x) = u$.

The following example shows the superiority of the main result in the sense that there exist a complete cone b-metric space over Banach algebra and a map $f : X \to X$ such that Corollary 1 is not applicable to, while our result is.

Example 2. *Let $\mathcal{A} = \mathbb{R}^2$, $P = \{(x, y) \in \mathcal{A} : x, y \geq 0\}$, and $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathcal{A}$. Define*

(a) *the norm of $\mathcal{A}$ by $\|(x_1, x_2)\| = |x_1| + |x_2|$;*

(b) *the multiplication of $\mathcal{A}$ by $xy = (x_1, x_2)(y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1)$;*

(c) *$X = [0, \infty)$ and define $d : X \times X \to \mathcal{A}$ by $d(x, y) = \big(|x - y|^2, 0\big)$ for all $x, y \in X$;*

(d) *$f : X \to X$, $f(x) = \frac{x}{x+1}$ for all $x \in X$;*

(e) *$\varphi : P \to P$, $\varphi(z_1, z_2) = \left(\frac{z_1}{z_1+1}, 0\right)$ for all $(z_1, z_2) \in P$.*

Then

(1) *$\mathcal{A}$ is a Banach algebra with the identity element $e = (1, 0)$ and $\theta = (0, 0)$;*

(2) *for all* $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathcal{A}$, $x \succeq y$ *if and only if* $x_1 \geq y_1$ *and* $x_2 \geq y_2$*;*

(3) $(X, \mathcal{A}, d, s)$ *is a complete cone* b*-metric space over Banach algebra with* $s = 2$*;*

(4) *there does not exist* $k \in \mathcal{A}$ *with* $\rho(k) < 1$ *such that the condition* (12) *holds;*

(5) *all assumptions of Lemma 6 hold.*

Proof. (1). See [11, Theorem 3.1].

(2). Since $P = \{(x, y) \in \mathcal{A} : x, y \geq 0\}$, for any $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathcal{A}$, we have $x \succeq y$ if and only if $(x_1 - y_1, x_2 - y_2) \in P$. It is equivalent to $x_1 \geq y_1$ and $x_2 \geq y_2$.

(3). For any $x, y, z \in X = [0, \infty)$ we have

- $d(x, y) = \big(|x - y|^2, 0\big) \succeq (0, 0)$. So $d(x, y) \succeq \theta$, and $d(x, y) = \theta$ if and only if $x = y$;
- $d(x, y) = \big(|x - y|^2, 0\big) = \big(|y - x|^2, 0\big) = d(y, x)$.

Since $|x - z|^2 \leq 2\big(|x - y|^2 + |y - z|^2\big)$, we have $\big(|x - z|^2, 0\big) \leq 2\big[\big(|x - y|^2, 0\big) + \big(|y - z|^2, 0\big)\big]$. It implies that

$$d(x, z) \preceq 2\big(d(x, y) + d(y, z)\big).$$

By the above, d is a cone b-metric on X with $s = 2$.

Now for any Cauchy sequence $\{x_n\}$ in X and for each $c = (c_1, c_2) \in \mathrm{int}P$ there exists $m_0 \in \mathbb{N}$ such that for all $n, m > m_0$ we have

$$d(x_n, x_m) = \big(|x_n - x_m|^2, 0\big) \ll (c_1, c_2) = c.$$

This implies that for each $c_1 > 0$, we have $|x_n - x_m| \leq (c_1)^{\frac{1}{2}}$ for all $n, m > m_0$. It implies that $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}$. So there exists $x \in \mathbb{R}$ such that $\lim\limits_{n\to\infty} |x_n - x| = 0$. Since $x_n \in X = \mathbb{R}_+$ for all n and $x_n \to x$ in $\mathbb{R}$, we have $x \in X$. This implies that for each $c = (c_1, c_2) \in \mathrm{int}P$, there exists $m_0 \in \mathbb{N}$ such that for all $n > m_0$ we have $|x_n - x| \leq (c_1)^{\frac{1}{2}}$. Therefore, we get that for all $n > m_0$

$$d(x_n, x) = \big(|x_n - x|^2, 0\big) \ll (c_1, c_2) = c.$$

This proves that $\{x_n\}$ convergent to x in $(X, \mathcal{A}, d, s)$. So $(X, \mathcal{A}, d, s)$ is complete.

By the above, $(X, \mathcal{A}, d, s)$ is a complete cone b-metric space over Banach algebra with $s = 2$.

(4). Firstly, we observe that for $k = (k_1, k_2) \in P$, by induction we have

$$k^n = (k_1, k_2)^n = (k_1^n, nk_2k_1^{n-1}).$$

It implies that

$$\rho(k) = \inf \|k^n\|^{\frac{1}{n}} = \inf \big(|k_1^n| + |nk_2k_1^{n-1}|\big)^{\frac{1}{n}}. \tag{13}$$

Then, if $\rho(k) < 1$, by (13), we get that $k_1 < 1$.

On the contrary, suppose that there exists $k = (k_1, k_2) \in P$ with $\rho(k) < 1$ such that

$$d\big(f(x), f(y)\big) \preceq kd(x, y)$$

for all $x, y \in X$. Then for all $x, y \in X$,

$$\left(\left|\frac{x}{x+1} - \frac{y}{y+1}\right|^2, 0\right) \preceq (k_1, k_2)\big(|x - y|^2, 0\big).$$

For $x \neq 0$ and $y = 0$ we have

$$\Big(\Big|\frac{x}{x+1}\Big|^2, 0\Big) \preceq (k_1, k_2)\big(|x|^2, 0\big).$$

It is equivalent to

$$\Big(\Big|\frac{x}{x+1}\Big|^2, 0\Big) \preceq \big(k_1|x|^2, k_2|x|^2\big).$$

This implies that

$$\frac{|x|^2}{(x+1)^2} \leq k_1|x|^2.$$

Hence for all $x \neq 0$, we have

$$\frac{1}{(x+1)^2} \leq k_1. \tag{14}$$

Letting $x \to 0^+$ in (14) we get $1 \leq k_1$. This contradicts to the above observation.

(5). • For any $z = (z_1, z_2), t = (t_1, t_2) \in P$ with $z \preceq t$, that is, $0 \leq z_1 \leq t_1$ and $0 \leq z_2 \leq t_2$. Then we have

$$\frac{z_1}{z_1+1} \leq \frac{t_1}{t_1+1}.$$

It implies that

$$\varphi(z) = \Big(\frac{z_1}{z_1+1}, 0\Big) \preceq \Big(\frac{t_1}{t_1+1}, 0\Big) = \varphi(t).$$

So, for all $z, t \in P$ with $z \preceq t$, we have $\varphi(z) \preceq \varphi(t)$.

• Now for any $z = (z_1, z_2) \in P$ we have by induction that $\varphi^n(z) = \Big(\frac{z_1}{nz_1+1}, 0\Big)$. It follows that

$$(0,0) \leq \Big(\frac{z_1}{nz_1+1}, 0\Big) \text{ and } \lim_{n\to\infty} \frac{z_1}{nz_1+1} = 0.$$

This implies that

$$\theta \preceq \varphi^n(z) \quad \text{and} \quad \lim_{n\to\infty} \varphi^n(z) = \theta.$$

Therefore, for each $c = (c_1, c_2) \in \mathrm{int}P$, by Lemma 4.(5) there exists $m_0 \in \mathbb{N}$ such that for all $n > m_0$ we have

$$\Big(\frac{z_1}{nz_1+1}, 0\Big) \ll (c_1, c_2) = c.$$

This implies that $\{\varphi^n(z)\}$ is a **c**-sequence in P.

• Suppose that $\{z_n\} = \{(z_1^{(n)}, z_2^{(n)})\}$ is a **c**-sequence in P, then for each $c = (c_1, c_2) \in \mathrm{int}P$, there exists $k_0 \in \mathbb{N}$ such that for all $n > k_0$ we have $(z_1^{(n)}, z_2^{(n)}) \ll (c_1, c_2) = c$. This implies that

$$\varphi(z_n) = \Big(\frac{z_1^{(n)}}{z_1^{(n)}+1}, 0\Big) \preceq (z_1^{(n)}, z_2^{(n)}) \ll (c_1, c_2) = c, \text{ for all } n > k_0.$$

Therefore, $\{\varphi(z_n)\}$ is also a **c**-sequence in P.

Hence φ is a weak comparison.

Next, for any $x, y \in X$, we have

$$\begin{aligned}\Big(\Big|\frac{x}{x+1} - \frac{y}{y+1}\Big|^2, 0\Big) &= \Big(\Big|\frac{x-y}{xy+x+y+1}\Big|^2, 0\Big) \leq \Big(\Big|\frac{|x-y|}{|x-y|+1}\Big|^2, 0\Big) \\ &= \Big(\Big|\frac{|x-y|^2}{|x-y|^2+2|x-y|+1}\Big|, 0\Big) \leq \Big(\frac{|x-y|^2}{|x-y|^2+1}, 0\Big).\end{aligned} \tag{15}$$

Note that

$$d\big(f(x), f(y)\big) = d\Big(\frac{x}{x+1}, \frac{y}{y+1}\Big) = \Big(\Big|\frac{x}{x+1} - \frac{y}{y+1}\Big|^2, 0\Big)$$

and

$$\varphi(d(x,y)) = \Big(\frac{|x-y|^2}{|x-y|^2+1}, 0\Big).$$

So from (15) we find that $d\big(f(x), f(y)\big) \preceq \varphi(d(x,y))$ for all $x, y \in X$, and f is a generalized weak φ-contraction.

By the above, all assumptions of Lemma 6 hold. $\square$

3 APPLICATIONS TO THE NONLINEAR INTEGRAL EQUATIONS

In this section, we apply Lemma 6 to study the existence and uniqueness of the solution to the nonlinear integral equations.

Lemma 7. *Let $C[a,b]$ be the set of all continuous functions on $[a,b]$, where $a, b \in \mathbb{R}$. Let $\mathcal{A} = \mathbb{R}^2$ and $P = \{(x,y) \in \mathcal{A} : x, y \geq 0\}$ with the same norm, the same multiplication, and the same partial order on $\mathcal{A}$ as stated in Example 2. Define $d : C[a,b] \times C[a,b] \to \mathcal{A}$ by*

$$d(x,y) = \Big(\sup_{t\in[a,b]} |x(t) - y(t)|^2, \sup_{t\in[a,b]} |x(t) - y(t)|^2\Big)$$

for all $x, y \in C[a,b]$. Then $\big(C[a,b], \mathcal{A}, d, s\big)$ is a complete cone b-metric space over Banach algebra with $s = 2$.

Proof. For any $x, y, z \in C[a,b]$ we have

$d(x,y) = \Big(\sup_{t\in[a,b]} |x(t) - y(t)|^2, \sup_{t\in[a,b]} |x(t) - y(t)|^2\Big) \geq (0,0)$. So $d(x,y) \succeq \theta$.

$d(x,y) = \theta$ if and only if $\sup_{t\in[a,b]} |x(t) - y(t)|^2 = 0$ if and only if $x(t) = y(t)$ for all $t \in [a,b]$, that is, $x = y$.

Since $\sup_{t\in[a,b]} |x(t) - y(t)|^2 = \sup_{t\in[a,b]} |y(t) - x(t)|^2$ for all $t \in [a,b]$, we get that $d(x,y) = d(y,x)$.

We have

$$|x(t) - z(t)|^2 \leq 2\big(|x(t) - y(t)|^2 + |y(t) - z(t)|^2\big) \text{ for all } t \in [a,b].$$

It implies that

$$\sup_{t\in[a,b]} |x(t) - z(t)|^2 \leq 2\Big(\sup_{t\in[a,b]} |x(t) - y(t)|^2 + \sup_{t\in[a,b]} |y(t) - z(t)|^2\Big) \text{ for all } t \in [a,b].$$

That is,

$$d(x,z) \preceq 2\big(d(x,y) + d(y,z)\big).$$

By the above, d is a cone b-metric on X with $s = 2$.

Now for any Cauchy sequence $\{x_n\}$ in $\big(C[a,b], \mathcal{A}, d, s\big)$ and for each $c = (c_1, c_2) \in \mathrm{int}P$, there exists $m_0 \in \mathbb{N}$ such that for all $n, m > m_0$ we have

$$d(x_n, x_m) = \big(\sup_{t\in[a,b]} |x_n(t) - x_m(t)|^2, \sup_{t\in[a,b]} |x_n(t) - x_m(t)|^2\big) \ll (c_1, c_2) = c. \tag{16}$$

This implies that

$$\sup_{t\in[a,b]} |x_n(t) - x_m(t)| \leq \sqrt{c_i},\ i = 1,2,\ \text{for all } n, m > m_0. \tag{17}$$

So $\{x_n\}$ is a Cauchy sequence in $C[a,b]$. Since $C[a,b]$ with the sup-norm is complete, there exists $x \in C[a,b]$ such that $\lim\limits_{n\to\infty} x_n = x$. Hence by (17) we have

$$\sup_{t\in[a,b]} |x_n(t) - x(t)| \leq \sqrt{c_i},\ i = 1,2,\ \text{for all } n > m_0.$$

This implies by (16) that

$$d(x_n, x) = \Big(\sup_{t\in[a,b]} |x_n(t) - x(t)|^2, \sup_{t\in[a,b]} |x_n(t) - x(t)|^2\Big) \ll (c_1, c_2) = c,\ \text{for all } n > m_0.$$

This proves that $\{x_n\}$ converges to x in $\big(C[a,b], \mathcal{A}, d, s\big)$. So $\big(C[a,b], \mathcal{A}, d, s\big)$ is complete.

By the above, $\big(C[a,b], \mathcal{A}, d, s\big)$ is a complete cone b-metric space over Banach algebra with $s = 2$. ☐

Theorem 2. *Let $\big(C[a,b], \mathcal{A}, d, s\big)$ be a complete cone b-metric space over Banach algebra in Lemma 7. Consider a integral equation*

$$x(t) = \eta(t) + \int_a^b K\big(t, x(r)\big)dr, \quad t \in [a,b], \tag{18}$$

where $x \in C[a,b]$, $\eta \in C[a,b]$ and $K : [a,b] \times \mathbb{R} \to \mathbb{R}$. Assume that the following hypotheses hold:

1) *for each $t \in [a,b]$, $K\big(t, x(r)\big)$ is integrable with respect to r on $[a,b]$;*

2) *there exists a continuous function $\psi : [a,b] \times [a,b] \to \mathbb{R}$ with $\sup\limits_{t\in[a,b]} \int_a^b \big|\psi(t,r)\big|dr \leq 1$ and there exists a comparison function γ such that for all $t, r \in [a,b]$ and all $x, y \in C[a,b]$,*

$$\Big|K\big(t, x(r)\big) - K\big(t, y(r)\big)\Big| \leq \big|\psi(t,r)\big|\gamma\big(|x(r) - y(r)|\big).$$

Then the integral equation (18) *has a unique solution $u \in C[a,b]$.*

Proof. Let $f : C[a,b] \to C[a,b]$ be a map defined by

$$\big(f(x)\big)(t) = \eta(t) + \int_a^b K\big(t, x(r)\big)dr, \quad x \in C[a,b],\ t \in [a,b].$$

For any $x, y \in C[a,b]$, we have

$$\begin{aligned}
& d\Big(\big(f(x)\big)(t), \big(f(y)\big)(t)\Big) \\
&= \Big(\sup_{t\in[a,b]} \big|\big(f(x)\big)(t) - \big(f(y)\big)(t)\big|^2, \sup_{t\in[a,b]} \big|\big(f(x)\big)(t) - \big(f(y)\big)(t)\big|^2\Big) \\
&= \Big(\sup_{t\in[a,b]} \Big|\int_a^b \Big[K\big(t,x(r)\big) - K\big(t,y(r)\big)\Big]dr\Big|^2, \sup_{t\in[a,b]} \Big|\int_a^b \Big[K\big(t,x(r)\big) - K\big(t,y(r)\big)\Big]dr\Big|^2\Big) \\
&\le \Big(\sup_{t\in[a,b]} \Big[\int_a^b \Big|K\big(t,x(r)\big) - K\big(t,y(r)\big)\Big|dr\Big]^2, \sup_{t\in[a,b]} \Big[\int_a^b \Big|K\big(t,x(r)\big) - K\big(t,y(r)\big)\Big|dr\Big]^2\Big) \\
&\le \Big(\sup_{t\in[a,b]} \Big[\int_a^b |\psi(t,r)|\gamma\big(|x(r) - y(r)|\big)dr\Big]^2, \sup_{t\in[a,b]} \Big[\int_a^b |\psi(t,r)|\gamma\big(|x(r) - y(r)|\big)dr\Big]^2\Big) \\
&\le \Big(\sup_{t\in[a,b]} \Big[\int_a^b |\psi(t,r)|\gamma\big(\sup_{r\in[a,b]} |x(r) - y(r)|\big)dr\Big]^2, \sup_{t\in[a,b]} \Big[\int_a^b |\psi(t,r)|\gamma\big(\sup_{r\in[a,b]} |x(r) - y(r)|\big)dr\Big]^2\Big) \\
&\le \Big(\gamma^2\big(\sup_{r\in[a,b]} |x(r) - y(r)|\big), \gamma^2\big(\sup_{r\in[a,b]} |x(r) - y(r)|\big)\Big) \\
&= \Big(\gamma^2\Big(\sqrt{\sup_{r\in[a,b]} |x(r) - y(r)|^2}\Big), \gamma^2\Big(\sqrt{\sup_{r\in[a,b]} |x(r) - y(r)|^2}\Big)\Big) \\
&= \varphi\Big(\sup_{r\in[a,b]} |x(r) - y(r)|^2\big), \sup_{r\in[a,b]} |x(r) - y(r)|^2\big)\Big) = \varphi\big(d(x,y)\big),
\end{aligned}$$

where $\varphi : P \to P$ defined by $\varphi(z) = \varphi(z_1, z_2) = \Big(\gamma^2(\sqrt{z_1}), \gamma^2(\sqrt{z_2})\Big)$ for all $z = (z_1, z_2) \in P$.

Now we prove that $\varphi(z)$ is a weak comparison.

- For any $z = (z_1, z_2)$, $t = (t_1, t_2) \in P$ with $z \preceq t$. Then we have $0 \le z_1 \le t_1$ and $0 \le z_2 \le t_2$. It follows that

$$0 \le \gamma(\sqrt{z_1}) \le \gamma(\sqrt{t_1}) \text{ and } 0 \le \gamma(\sqrt{z_2}) \le \gamma(\sqrt{t_2}).$$

This implies that

$$\gamma^2(\sqrt{z_1}) \le \gamma^2(\sqrt{t_1}) \text{ and } \gamma^2(\sqrt{z_2}) \le \gamma^2(\sqrt{t_2}).$$

Therefore, we get

$$\varphi(z) = \Big(\gamma^2(\sqrt{z_1}), \gamma^2(\sqrt{z_2})\Big) \preceq \Big(\gamma^2(\sqrt{t_1}), \gamma^2(\sqrt{t_2})\Big) = \varphi(t).$$

So, for all $z, t \in P$ with $z \preceq t$, we have $\varphi(z) \preceq \varphi(t)$.

- Since $z = (z_1, z_2) \in P$ and γ is the comparison function, we have

$$(0,0) \le \Big(\gamma^{2n}(\sqrt{z_1}), \gamma^{2n}(\sqrt{z_2})\Big) \text{ and } \lim_{n\to\infty} \gamma^{2n}(\sqrt{z_1}) = \lim_{n\to\infty} \gamma^{2n}(\sqrt{z_2}) = 0.$$

This implies that

$$\theta \preceq \varphi^n(z) \text{ and } \lim_{n\to\infty} \varphi^n(z) = \theta.$$

Therefore, for each $c = (c_1, c_2) \in \mathrm{int}P$, by Lemma 4.(5) there exists $m_0 \in \mathbb{N}$ such that for all $n > m_0$ we have

$$\varphi^n(z) \ll (c_1, c_2) = c.$$

This prove that $\{\varphi^n(z)\}$ is a **c**-sequence in P.

• Suppose that $\{z_n\} = \{(z_1^{(n)}, z_2^{(n)})\}$ is a **c**-sequence in P, then for each $c = (c_1, c_2) \in \mathrm{int}P$, there exists $l_0 \in \mathbb{N}$ such that for all $n > l_0$ we have $(z_1^{(n)}, z_2^{(n)}) \ll (c_1, c_2)$. Since γ is a comparison function, we find that

$$\begin{aligned}\varphi(z_n) &= \varphi(z_1^{(n)}, z_2^{(n)}) = \Big(\gamma^2\big(\sqrt{z_1^{(n)}}\big), \gamma^2\big(\sqrt{z_2^{(n)}}\big)\Big)\\ &\preceq \Big(\gamma\big(\sqrt{z_1^{(n)}}\big), \gamma\big(\sqrt{z_2^{(n)}}\big)\Big) \preceq \big(\sqrt{z_1^{(n)}}, \sqrt{z_2^{(n)}}\big) \ll (\sqrt{c_1}, \sqrt{c_2}).\end{aligned}$$

This implies that $\{\varphi(z_n)\}$ is also a **c**-sequence in P. Hence φ is a weak comparison.

Thus, all the conditions of Lemma 6 hold, and hence the integral equation (18) has a unique solution $u \in C[a,b]$. □

The following example guarantees the existence of the function K, ψ, γ and η that satisfies all assumptions in Theorem 2.

Example 3. *Let $C[0,1]$ be the set of all continuous functions on $[0,1]$. Consider the nonlinear integral equation*

$$x(t) = t - \Big(\frac{3}{4} + \ln\frac{2\sqrt{6}}{9}\Big).\sin t + \int_0^1 r.\sin t.\ln\big(1 + \frac{1}{2}|x(r)|\big)dr, \quad t \in [0,1]. \tag{19}$$

Put

$$\eta(t) = t - \Big(\frac{3}{4} + \ln\frac{2\sqrt{6}}{9}\Big).\sin t, \quad \psi(t,r) = r.\sin t \quad \textit{for all } t, r \in [0,1],$$

and

$$K\big(t, x(r)\big) = r.\sin t.\ln\big(1 + \frac{1}{2}|x(r)|\big) \quad \text{for all } x \in C[0,1] \quad \text{and all } t, r \in [0,1].$$

Then

(1) *$\eta \in C[0,1]$ and $K\big(t, x(r)\big)$ is integrable with respect to r on $[0,1]$;*

(2) *$\psi(t,r)$ is continuous on $[0,1]\times[0,1]$ and $\sup\limits_{t\in[0,1]} \int_0^1 \mid \psi(t,r) \mid dr < 1$;*

(3) *put $\gamma(u) = \ln(1 + \frac{1}{2}u)$ for all $u \in \mathbb{R}_+$, we have γ is a comparison function;*

(4) *for all $t, r \in [0,1]$ and $x, y \in C[0,1]$, we have*

$$\big|K\big(t, x(r)\big) - K\big(t, y(r)\big)\big| \le \big|\psi(t,r)\big|\gamma\big(|x(r) - y(r)|\big).$$

Proof. (1). Since $\eta(t) = t - \Big(\frac{3}{4} + \ln\frac{2\sqrt{6}}{9}\Big).\sin t$ for all $t \in [0,1]$, we have $\eta \in C[0,1]$. Since $x \in C[0,1]$, we have $K\big(t, x(r)\big) = r.\sin t.\ln\big(1 + \frac{1}{2}|x(r)|\big)$ is integrable with respect to r on $[0,1]$.

(2). It is easy to see that $\psi(t,r)$ is continuous on $[0,1]\times[0,1]$ and $\sup\limits_{t\in[0,1]} \int_0^1 \mid \psi(t,r) \mid dr < 1$.

(3). For all $u_1, u_2 \in \mathbb{R}_+$ and $u_1 \le u_2$, we have $\gamma(u_1) = \ln(1 + \frac{1}{2}u_1) \le \ln(1 + \frac{1}{2}u_2) = \gamma(u_2)$.

For any $u \in \mathbb{R}_+$, we have

$$\gamma(u) = \ln(1+\frac{1}{2}u) \leq \frac{1}{2}u$$

and

$$\gamma^2(u) = \gamma\big(\gamma(u)\big) = \ln\big(1+\frac{1}{2}\ln(1+\frac{1}{2}u)\big) \leq \frac{1}{2}\ln(1+\frac{1}{2}u) \leq \frac{1}{2^2}u.$$

Continuing the above process we obtain that for all n,

$$\gamma^n(u) \leq \frac{1}{2^n}u.$$

From the above, we have γ is increasing and $\lim\limits_{n\to\infty}\gamma^n(u) = 0$.

(4). Now let $x, y \in C[0,1]$. Then, for each $r, t \in [0,1]$, we have

$$\begin{aligned}
\mid K\big(t, x(r)\big) - K\big(t, y(r)\big) \mid &= \Big| r.\sin t.\ln\big(1+\frac{1}{2}|x(r)|\big) - r.\sin t.\ln\big(1+\frac{1}{2}|y(r)|\big)\Big| \\
&= |r.\sin t|.\Big|\ln\big(1+\frac{1}{2}|x(r)|\big) - \ln\big(1+\frac{1}{2}|y(r)|\big)\Big| \\
&= |r.\sin t|.\Big|\ln\Big(\frac{1+\frac{1}{2}|x(r)|}{1+\frac{1}{2}|y(r)|}\Big)\Big| \\
&= |r.\sin t|.\Big|\ln\Big(1+\frac{\frac{1}{2}|x(r)| - \frac{1}{2}|y(r)|}{1+\frac{1}{2}|y(r)|}\Big)\Big| \\
&\leq |r.\sin t|.\Big|\ln\Big(1+\frac{\frac{1}{2}|x(r)-y(r)|}{1+\frac{1}{2}|y(r)|}\Big)\Big| \\
&\leq |r.\sin t|.\Big|\ln\Big(1+\frac{1}{2}\big|x(r)-y(r)\big|\Big)\Big| \\
&= \big|\psi(t,r)\big|.\gamma\big(\big|x(r)-y(r)\big|\big).
\end{aligned}$$

From the above, K, ψ, γ and η satisfy all assumptions of Theorem 2. Hence the integral equation (19) has a unique solution $u \in C[0,1]$. □

REFERENCES

[1] Aydi H., Karapinar E., Mustafa Z. *Coupled coincidence point results on generalized distance in ordered cone metric spaces*. Positivity. 2013, **17** (4), 979–993. doi:10.1007/s11117-012-0216-2

[2] Aydi H., Nashine H. K., Samet B., Yazidi H. *Coincidence and common fixed point results in partially ordered cone metric spaces and applications to integral equations*. Nonlinear Anal. 2011, **74** (17), 6814–6825. doi:10.1016/j.na.2011.07.006

[3] Van Dung N., Le Hang V. T. *Remarks on cyclic contractions in b-metric spaces and applications to integral equations*. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM 2017, **111** (1), 247–255. doi:10.1007/s13398-016-0291-5

[4] Dordević M., Doric D., Kadelburg Z., Radenović S., Spasic D. *Fixed point results under c-distance in tvs-cone metric spaces*. Fixed Point Theory Appl. 2011, **2011**, 1–9. doi:10.1186/1687-1812-2011-29

[5] Guang H. L., Xian Z. *Cone metric spaces and fixed point theorems of contractive mappings*. J. Math. Anal. Appl. 2007, **332**, 1468–1476. doi:10.1016/j.jmaa.2005.03.087

[6] Huang H., Radenović S. *Common fixed point theorems of generalized Lipschitz mappings in cone b-metric spaces over Banach algebras and applications*. J. Nonlinear Sci. Appl. 2015, **8**, 787–799.

[7] Huang H. Radenović S. *Some fixed point results of generalized Lipschitz mappings on cone b-metric spaces over Banach algebras*. J. Comput. Anal. Appl. 2016, **20** (3), 566–583.

[8] Huang H., Radenovic S., Deng G. *A sharp generalization on cone b-metric space over Banach algebra*. J. Nonlinear Sci. Appl. 2017, **10** (2), 429–435. doi:10.22436/jnsa.010.02.09

[9] Hussaina N., Shahb M. H. *KKM mappings in cone b-metric spaces*. Comput. Math. Appl. 2011, **62**, 1677–1684.

[10] Jankovic S., Kadelburg Z., Radenović S. *On cone metric spaces: a survey*. Nonlinear Anal. 2011, **74** (7), 2591–2601. doi:10.1016/j.na.2010.12.014

[11] Li B., Huang H. *Fixed point results for weak φ-contractions in cone metric spaces over Banach algebras and applications*. J. Funct. Spaces 2017, **2017** (5), 1–6. doi:10.1155/2017/5054603

[12] Liu H., Xu S. *Cone metric spaces over Banach algebras and fixed point theorems of generalized Lipschitz mappings*. Fixed Point Theory Appl. 2013, **2013**, 1–10. doi:10.1186/1687-1812-2013-320

[13] Rezapour Sh., Hamlbarani R. *Some notes on the paper Cone metric spaces and fixed point theorems of contractive mappings*. Math. Anal. Appl. 2008, **345** (2), 719–724. doi:10.1016/j.jmaa.2008.04.049

[14] Rudin W. Functional Analysis, McGraw-Hill, New York, NY, USA, 2nd edition, 1991.

[15] Rus I. A., Şerban M. A. *Basic problems of the metric fixed point theory and the relevance of a metric fixed point theorem*. Carpathian J. Math. 2013, **29** (2), 239–258.

[16] Xu S., Radenović S. *Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebras without assumption of normality*. Fixed Point Theory Appl. 2014, **2014**, 1–12. doi:10.1186/1687-1812-2014-102

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Кван Л.Т., Ван Ан Т. *Про розв'язки деякого класу нелінійних інтегральних рівняннь в конічних b-метричних просторах над банаховими алгебрами* // Карпатські матем. публ. — 2019. — Т.11, №1. — C. 163–178.

У даній роботі ми вивчаємо існування розв'язків деякого класу функціональних інтегральних рівнянь з використанням деяких результатів про фіксовану точку у конічних b-метричних просторах над банаховими алгебрами. Для отримання цих результатів ми ввели і довели деякі властивості узагальнених слабких φ-скорочень, в яких φ є нелінійними слабкими функціями порівняння. Отримані результати є узагальненнями результатів Van Dung N., Le Hang V. T., Huang H., Radenovic S. і Deng G. Також, наведено деякі відповідні приклади для ілюстрації отриманих результатів.

Ключові слова і фрази: конічний b-метричний простір над банаховою алгеброю, φ-скорочення, **c**-послідовність, нерухома точка, інтегральне рівняння.