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PARABOLIC SYSTEMS OF SHILOV-TYPE WITH COEFFICIENTS OF BOUNDED SMOOTHNESS AND NONNEGATIVE GENUS

The Shilov-type parabolic systems are parabolically stable systems for changing its coefficients unlike of parabolic systems by Petrovskii. That's why the modern theory of the Cauchy problem for class by Shilov-type systems is developing abreast how the theory of the systems with constant or time-dependent coefficients alone. Building the theory of the Cauchy problem for systems with variable coefficients is actually today. A new class of linear parabolic systems with partial derivatives to the first order by the time t with variable coefficients that includes a class of the Shilov-type systems with time-dependent coefficients and non-negative genus is considered in this work. A main part of differential expression concerning space variable x of each such system is parabolic (by Shilov) expression. Coefficients of this expression are time-dependent, but coefficients of a group of younger members may depend also a space variable. We built the fundamental solution of the Cauchy problem for systems from this class by the method of sequential approximations. Conditions of minimal smoothness on coefficients of the systems by variable x are founded, the smoothness of solution is investigated and estimates of derivatives of this solution are obtained. These results are important for investigating of the correct solution of the Cauchy problem for this systems in different functional spaces, obtaining forms of description of the solution of this problem and its properties.

Key words and phrases: fundamental matrix of solutions, Cauchy problem, Shilov-type parabolic systems.

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INTRODUCTION

The definition of parabolicity formulated by G.Ye. Shilov [12] generalizes the definition of parabolicity by I.G. Petrovskii [11] and extends considerably the Petrovskii's class of the first-order on time systems by the systems with constant coefficients with order different form the parabolicity factor. The parabolic (by Shilov) systems were investigated, in part, in papers [2, 4, 6, 7] containing the results on description of the classes of uniqueness and correctness of the Cauchy problem, developing the methods of study of fundamental solution, rating the correct solvability of the Cauchy problem at various functional spaces, and ascertaining qualitative properties of solutions for such systems. However, these results concern to the systems with constant or time-dependent coefficients alone. The attempts to derive any results for parabolic (by Shilov) systems with variable coefficients, which are space-dependent ones, were unsuccessful, while it has been shown [5] that such systems are parabolically unstable to changing coefficients.

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Interesting approach to expansion of the Shilov class of parabolic systems has been proposed by Ya.I. Zhitomirskii [13] defining a new class of parabolically stable systems to variations of the lower coefficients. This class adds naturally to the Petrovskii's class of systems with variable coefficients and covers the parabolic (by Shilov) systems. These systems are referred to as the Shilov-type parabolic systems with variable coefficients.

The Shilov-type parabolic systems of the p-th order are of the form

$$\partial_t u(t;x) = \{P_0(t;i\partial_x) + P_1(t,x;i\partial_x)\}u(t;x), \quad t \in (0;T],\ x \in \mathbb{R}^n, \tag{1}$$

where $u := \mathrm{col}(u_1,\dots,u_m)$, $P_0(t;i\partial_x)$ and $P_1(t,x;i\partial_x)$ are the matrix differential expressions of the orders p and p_1, respectively, with coefficients dependent on time t, and for P_1 on spatial variable x as well. For that, the system

$$\partial_t u(t;x) = P_0(t;i\partial_x)u(t;x), \quad t \in (0;T],\ x \in \mathbb{R}^n, \tag{2}$$

is the parabolical (by Shilov) system with the parabolicity factor h, $0 < h \le p$, kind of μ and of reduced order p_0 (see [4, p.72, p.133]), and p_1 satisfies the following conditions:

$$(A) \qquad 0 \le p_1 < h - n\Big(1 - h\mu/p_0\Big) - (m-1)(p-h), \quad 0 \le \mu;$$
$$(\widehat{A}) \qquad 0 \le p_1 < h - n(1-\mu) - (m-1)(p-h), \quad \mu < 0.$$

For the systems (1) Ya.I. Zhitomirskii has ascertained by the method of sequential approximations correct solvability of the Cauchy problem at the class of smooth bounded initial functions for the case, when the coefficients of the differential expression for P_0 are constant, and the coefficients of the expression P_1 are limited being dependent on x, alone functions, which are differentiable up to some order.

Further elaboration of the Cauchy problem for the Shilov-type parabolic systems with variable coefficients presumed construction of the fundamental solution of the Cauchy problem (FSCP) and comprehensive investigation of it.

For the systems (1) of nonnegative kind μ and the coefficients, which are boundedly continuous on t and infinitely differentiable on x, the FSCP has been derived and its main properties have been studied [8]. These results enable to develop the theory of the Cauchy problem [1, 9, 10] for such systems at spaces S of I.M. Gelfand and G.Ye. Shilov and, in part, to prove correct solvability of the Cauchy problem with generalized initial conditions of kind of the Gevrey's ultra-distributions, to find out the form of classical solutions with generalized boundary values at initial hyperplane, to study the properties of localization and stabilization of the solutions, and to describe the sets of generalized initial functions for which the corresponding solutions are the elements of the L. Swartz space S or any of spaces of I.M. Gelfand and G.Ye. Shilov.

In this paper, we continue the study of the systems (1) for $\mu \ge 0$ with coefficients of bounded smoothness. We determine the conditions of minimal smoothness of the coefficients with respect to the variable x, for which the classical FSCP exists, construct this solution and investigate its main properties. These results are important for further development of the classical theory of the Cauchy problem for parabolic systems and its unification.

1 Auxiliary data

Let T be a fixed number from $(0;+\infty)$, $\mathbb{N}$ be the set of natural numbers; $\mathbb{N}_m := \{1,\dots,m\}$; $\mathbb{R}^n$ be the real n-dimension space; $\mathbb{R} := \mathbb{R}^1$; $\mathbb{Z}_+^n$ be the set of all n-dimension multi-indices,

$\mathbb{Z}_+ := \mathbb{Z}^1_+$; i – imaginary unit; $(\cdot,\cdot)$ – scalar product at the space $\mathbb{R}^n$; $\|x\| := (x,x)^{1/2}$, $x \in \mathbb{R}^n$; $|x+iy| := (x^2+y^2)^{1/2}$, $\{x,y\} \subset \mathbb{R}$; $|(a_{lj})^m_{l,j=1}| := \max\limits_{\{l,j\}\subset\mathbb{N}_m} |a_{lj}|$; $|z|_+ := |z_1| + \ldots + |z_n|$, $z^l := z_1^{l_1}\ldots z_n^{l_n}$, if $z \in \mathbb{R}^n$, $l \in \mathbb{Z}^n_+$; $\Pi_M := \{(t;x)|\ t \in M, x \in \mathbb{R}^n\}$, $M \subset \mathbb{R}$.

We will consider here only the systems (1) with $\mu \geq 0$, where the differential expressions for P_0 and P_1 are of the form

$$P_0(t;i\partial_x) = \sum_{|k|_+\leq p} A_{0,k}(t)\partial_x^k,\ \ P_1(t,x;i\partial_x) = \sum_{|k|_+\leq p_1} A_{1,k}(t;x)\partial_x^k,$$

where $A_{0,k}(t) := i^{|k|_+}\left(a^{lj}_{0,k}(t)\right)^m_{l,j=1}$, $A_{1,k}(t;x) := i^{|k|_+}\left(a^{lj}_{1,k}(t;x)\right)^m_{l,j=1}$ are matrix coefficients.

By $G(t,\tau;\cdot)$, $0 \leq \tau < t \leq T$, we denote FSCP of system (2). It is known that $G(t,\tau;\cdot) = F\left[\Theta^t_\tau(\xi)\right](t,\tau;\cdot)$, where $F[\cdot]$ is the Fourier transformation operator, and $\Theta^t_\tau(\cdot)$ is a matriciant of the corresponding Fourier duality of the system. The following statement is proper [1,6].

Proposition 1.1. *For all $T > 0$ there exists $\delta > 0$ and for all $k \in \mathbb{Z}^n_+$ there exists $c > 0$ such that for all $t \in (\tau;T]$, $\tau \in [0;T)$ and $\{x,\xi\} \subset \mathbb{R}^n$ takes place*

$$|\partial_x^k G(t,\tau;x-\xi)| \leq c(t-\tau)^{-\frac{n+|k|_++\gamma}{h}} e^{-\delta\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}, \tag{3}$$

where $\gamma := (m-1)(p-h)$ and $\alpha := \mu/p_0$.

Here, we consider systems (1), which satisfy, in addition to condition (A), the following condition:

(B) the coefficients $a^{lj}_{0,k}(t)$, $a^{lj}_{1,k}(t;x)$ are continuous in the variable t uniformly with respect to x, differentiable with respect to the variable x up to the order α_* inclusively, and bounded together with their derivatives by complex-valued functions in a ball $\Pi_{[0;T]}$.

In [8], FSCP of system (1) was constructed in the form

$$Z(t,x;\tau,\xi) = G(t,\tau;x-\xi) + W(t,x;\tau,\xi),\ \ (t,x;\tau,\xi) \in \Pi^2_T, \tag{4}$$

where $\Pi^2_T := \{(t,x;\tau,\xi)|\ 0 \leq \tau < t \leq T, \{x,\xi\} \subset \mathbb{R}^n\}$ and

$$W(t,x;\tau,\xi) := \int_\tau^t d\beta \int_{\mathbb{R}^n} G(t,\beta;x-y)\Phi(\beta,y;\tau,\xi)dy. \tag{5}$$

Here

$$\Phi(t,x;\tau;\xi) = \sum_{l=1}^\infty K_l(t,x;\tau,\xi), \tag{6}$$

where

$$\begin{aligned} &K_1(t,x;\tau,\xi) := P_1(t,x;i\partial_x)G(\tau,t;x-\xi),\\ &K_l(t,x;\tau,\xi) := \int_\tau^t d\beta \int_{\mathbb{R}^n} K_1(t,x;\beta,y)K_{l-1}(\beta,y;\tau,\xi)dy,\ l > 1. \end{aligned} \tag{7}$$

In this case, it was established that condition (A) and the boundedness of the coefficients of system (1) ensure the absolute uniform convergence of the functional series (6) for all $\{x,\xi\} \subset$

$\mathbb{R}^n$, $t \in (\tau; T]$, and $\tau \in [0, T)$. Moreover, its sum Φ and the iterated kernels K_l satisfy the estimates

$$|\Phi(t,x;\tau,\xi)| \le c_1 (t-\tau)^{\alpha_0-(1+\alpha n)} e^{-\delta_1 \left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}, \tag{8}$$

$$\begin{aligned} |K_l(t,x;\tau,\xi)| \le c_0^l \left(\prod_{j=1}^{l-1} c_{(j\varepsilon)} B(\alpha_0, j\alpha_0)\right) \\ \times (t-\tau)^{l\alpha_0-(1+\alpha n)} e^{-\delta(1-(l-1)\varepsilon)\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}, \ \varepsilon \in (0;1), \end{aligned} \tag{9}$$

with the estimating constants independent of t, τ, x, and ξ. Here

$$\alpha_0 := 1 + \alpha n - (n + p_1 + \gamma)/h > 0$$

and $B(\cdot,\cdot)$ is the Euler beta-function.

We note that estimates (3) and (8) for $\{x,\xi\} \subset \mathbb{R}^n$ and $0 \le \tau < t \le T$ guarantee the absolute convergence of the integral, by which the potential W is determined. Thus, the matrix function $Z(t,x;\tau,\xi)$ is properly determined by formula (4) on the whole set Π_T^2.

Completing this item, we present the following estimates from [3], which will be of importance in what follows:

$$e^{-\delta\left\{\left(\frac{\|x-y\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\|y-\xi\|}{(\beta-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}\right\}} \le e^{-\delta\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}; \tag{10}$$

$$\int_{\mathbb{R}^n} e^{-\delta\left\{\left(\frac{\|x-y\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\|y-\xi\|}{(\beta-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}\right\}} \frac{dy}{((t-\beta)(\beta-\tau))^{\alpha n}} \le \frac{c_\varepsilon e^{-\delta(1-\varepsilon)\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}}{(t-\tau)^{\alpha n}}, \quad \delta > 0, \tag{11}$$

(here, $\{x,y,\xi\} \subset \mathbb{R}^n$, $\beta \in (\tau;t)$, $0 \le \tau < t \le T$, $\varepsilon \in (0;1)$, and $\delta > 0$, and the quantity c_ε depends only on ε).

2 Properties of FSPC

First, we estimate the derivatives of the iterated kernels K_l.

According to representation (7), the smoothness of the kernel $K_1(t,x;\tau,\xi)$ in the spatial variables x and ξ is determined, respectively, by the smoothness of the coefficients of system (1) and the function $G(t,\tau;x-\xi)$. Therefore, there exist the derivatives $\partial_\xi^r \partial_x^q K_1$ for $\{r,q\} \subset \mathbb{Z}_+^n$, $|q|_+ \le \alpha_*$, and the following equality holds:

$$\partial_\xi^r \partial_x^q K_1(t,x;\tau,\xi) = \sum_{|k|_+ \le p_1} \sum_{l=0}^{q} C_q^l \left(\partial_x^l A_{1,k}(t;x)\right) \left(\partial_{(x-\xi)}^{k+r+q-l} G(t,\tau;x-\xi)\right),$$

where C_q^l is a binomial coefficient. From whence, with regard for condition (B) and estimate (3) for $\{r,q\} \subset \mathbb{Z}_+^n$, $|q|_+ \le \alpha_*$, $(t,x;\tau,\xi) \in \Pi_T^2$, we get

$$|\partial_\xi^r \partial_x^q K_1(t,x;\tau,\xi)| \le c_{r,q} (t-\tau)^{-\frac{n+p_1+\gamma+|r+q|_+}{h}} e^{-\delta\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}} \tag{12}$$

(here, the estimating constants are independent of t, τ, x, and ξ).

For $l > 1$, we will use the representation

$$\begin{aligned} K_l(t,x;\tau,\xi) = \int_{\tau}^{t_1} d\beta \int_{\mathbb{R}^n} K_1(t,x;\beta,\eta+\xi)K_{l-1}(\beta,\eta+\xi;\tau,\xi)d\eta \\ + \int_{t_1}^{t} d\beta \int_{\mathbb{R}^n} K_1(t,x;\beta,x-z)K_{l-1}(\beta,x-z;\tau,\xi)dz, \ t_1 := \frac{t+\tau}{2}. \end{aligned} \tag{13}$$

According to it,

$$\begin{aligned} \partial_\xi^r \partial_x^q K_l(t,x;\tau,\xi) = \sum_{|r_1|_+ \le |r|_+} C_r^{r_1} \int_{\tau}^{t_1} d\beta \int_{\mathbb{R}^n} \Big(\partial_\xi^{r_1}\partial_x^q K_1(t,x;\beta,\eta+\xi)\Big) \\ \times \Big(\partial_\xi^{r-r_1} K_{l-1}(\beta,\eta+\xi;\tau,\xi)\Big)\, d\eta + \sum_{|q_1|_+ \le |q|_+} C_q^{q_1} \int_{t_1}^{t} d\beta \int_{\mathbb{R}^n} \big(\partial_x^{q_1} K_1(t,x;\beta,x-z)\big) \\ \times \Big(\partial_\xi^r \partial_x^{q-q_1} K_{l-1}(\beta,x-z;\tau,\xi)\Big)\, dz, \ |q|_+ \le \alpha_*, (t,x;\tau,\xi) \in \Pi_T^2. \end{aligned} \tag{14}$$

Hence, the estimation of $|\partial_\xi^r \partial_x^q K_l(t,x;\tau,\xi)|$ is reduced to that of the expressions $|\partial_\xi^r \partial_x^q K_1(t,x;\tau,\eta+\xi)|$, $|\partial_x^q K_1(t,x;\tau,x-z)|$, $|\partial_\xi^r K_{l-1}(t,\eta+\xi;\tau,\xi)|$, $|\partial_\xi^r \partial_x^q K_{l-1}(t,x-z;\tau,\xi)|$.

In view of the boundedness of $\partial_x^q a_{1,k}^{lj}(t;x)$, $|q|_+ \le \alpha_*$, and estimate (3), for all $\{q,r\} \in \mathbb{Z}_+^n$, $|q|_+ \le \alpha_*$, $\{x,\eta,\xi\} \in \mathbb{R}^n$, $t \in (\tau;T]$, and $\tau \in [0;T)$, we have

$$\begin{aligned} \big|\partial_\xi^r \partial_x^q K_1(t,x;\tau,\eta+\xi)\big| \le m \sum_{|k|_+ \le p_1} \sum_{|q_1|_+ \le |q|_+} C_q^{q_1} \big|\partial_x^{q_1} A_{1,k}(t;x)\big| \, \big|\partial_{(x-\eta-\xi)}^{k+r+q-q_1} G(t,\tau;x-\eta-\xi)\big| \\ \le c_{r,q}(t-\tau)^{-\frac{n+p_1+\gamma+|r+q|_+}{h}} e^{-\delta\left(\frac{\|x-\eta-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}; \end{aligned} \tag{15}$$

$$\begin{aligned} \big|\partial_x^q K_1(t,x;\tau,x-\xi)\big| = \Big|\partial_x^q \Big(\sum_{|k|_+ \le p_1} A_{1,k}(t;x)\partial_x^k G(t,\tau;\xi)\Big)\Big| \le m \Big|\partial_x^q A_{1,0}(t;x)\Big| \, \Big|G(t,\tau;\xi)\Big| \\ \le \widehat{c}_q(t-\tau)^{-\frac{n+\gamma}{h}} e^{-\delta\left(\frac{\|\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}} \le c_q(t-\tau)^{-\frac{n+p_1+\gamma}{h}} e^{-\delta\left(\frac{\|\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}. \end{aligned} \tag{16}$$

We now estimate the expression $\big|\partial_\xi^r K_l(t,\eta+\xi;\tau,\xi)\big|$. Since

$$\partial_\xi^r K_1(t,\eta+\xi;\tau,\xi) = \sum_{|k|_+ \le p_1} \partial_\xi^r A_{1,k}(t;\eta+\xi)\partial_\eta^k G(t,\tau;\eta), \ (t,x;\tau,\xi) \in \Pi_T^2, \tag{17}$$

we have, according to condition (B), that the iterated kernels $K_l(t,\eta+\xi;\tau,\xi)$ are differentiable with respect to the variable ξ only to the order α_*. This fact and (14) imply that $\partial_x^q K_l(t,x;\tau,\xi)$, $|q|_* \le \alpha_*$, is also a function differentiable with respect to ξ only to this order α_*.

Representation (17) and estimate (3) yield

$$\big|\partial_\xi^r K_1(t,\eta+\xi;\tau,\xi)\big| \le c_{1,r}(t-\tau)^{-\frac{n+p_1+\gamma}{h}} e^{-\delta\left(\frac{\|\eta\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}. \tag{18}$$

We note that

$$\partial_{\xi}^{r}K_2(t,\eta+\xi;\tau,\xi)=\partial_{\xi}^{r}\bigg(\int_{\tau}^{t}d\beta\int_{\mathbb{R}^n}K_1(t,\eta+\xi;\beta,y)K_1(\beta,y;\tau,\xi)dy\bigg).$$

Let us change the order of integration in the last integral by the formula $y=z+\xi$. In view of estimates (18) and (11) and the equalities

$$\int_{\tau}^{t}((t-\beta)(\beta-\tau))^{\alpha_0-1}d\beta=(t-\tau)^{2\alpha_0-1}B(\alpha_0,\alpha_0) \qquad (19)$$

and

$$\partial_{\xi}^{r}K_1(t,\eta+\xi;\tau,z+\xi)=\partial_{\zeta}^{r}K_1(t,(\eta-z)+\zeta;\tau,\zeta)\Big|_{\zeta=z+\xi},$$

we get

$$\begin{aligned}
&\big|\partial_{\xi}^{r}K_2(t,\eta+\xi;\tau,\xi)\big|\\
&\quad\le m\sum_{|r_1|_+\le|r|_+}C_r^{r_1}\int_{\tau}^{t}d\beta\int_{\mathbb{R}^n}\Big|\partial_{\xi}^{r_1}K_1(t,\eta+\xi;\beta,z+\xi)\Big|\,\Big|\partial_{\xi}^{r-r_1}K_1(\beta,z+\xi;\tau,\xi)\Big|dz\\
&\quad\le m\sum_{|r_1|_+\le|r|_+}C_r^{r_1}c_{1,r_1}c_{1,(r-r_1)}\int_{\tau}^{t}\big((t-\beta)(\beta-\tau)\big)^{-\frac{n+p_1+\gamma}{h}}\int_{\mathbb{R}^n}e^{-\delta\Big(\big(\frac{\|\eta-z\|}{(t-\beta)^{\alpha}}\big)^{\frac{1}{1-\alpha}}+\big(\frac{\|z\|}{(\beta-\tau)^{\alpha}}\big)^{\frac{1}{1-\alpha}}\Big)}dzd\beta\\
&\quad\le c_{2,r}(\varepsilon)B(\alpha_0,\alpha_0)(t-\tau)^{\alpha_0-\frac{n+p_1+\gamma}{h}}e^{-\delta(1-\varepsilon)\big(\frac{\|\eta\|}{(t-\tau)^{\alpha}}\big)^{\frac{1}{1-\alpha}}},\ \ \varepsilon\in(0;1).
\end{aligned} \qquad (20)$$

By reasoning analogously step by step, we arrive at the inequality

$$\big|\partial_{\xi}^{r}K_l(t,\eta+\xi;\tau,\xi)\big|\le c_{l,r}(\varepsilon)\Big(\prod_{j=1}^{l-1}B(\alpha_0,j\alpha_0)\Big)(t-\tau)^{(l-1)\alpha_0-\frac{n+p_1+\gamma}{h}}e^{-\delta(1-(l-1)\varepsilon)\big(\frac{\|\eta\|}{(t-\tau)^{\alpha}}\big)^{\frac{1}{1-\alpha}}}, \qquad (21)$$

which is satisfied for all $\{\eta,\xi\}\subset\mathbb{R}^n$, $|r|_+\le\alpha_*$, $0\le\tau<t\le T$, $\varepsilon\in(0;1)$, and $l\in\mathbb{N}\setminus\{1\}$ and, hence, until the existence of such number l_*, for which

$$\big|\partial_{\xi}^{r}K_{l_*}(t,\eta+\xi;\tau,\xi)\big|\le c_{l_*,r}(\varepsilon)\Big(\prod_{j=1}^{l_*-1}B(\alpha_0,j\alpha_0)\Big)e^{-\delta(1-(l_*-1)\varepsilon)\big(\frac{\|\eta\|}{(t-\tau)^{\alpha}}\big)^{\frac{1}{1-\alpha}}} \qquad (22)$$

(here, the quantities $c_{l,r}(\varepsilon)>0$ do not depend on the variables t,τ,η, and ξ, which vary in the above-indicated way).

Since

$$\partial_{\xi}^{r}\partial_{x}^{q}K_l(t,x;\tau,\eta+\xi)=\partial_{\zeta}^{r}\partial_{x}^{q}K_l(t,x;\tau,\zeta)\Big|_{\zeta=\eta+\xi}$$

and

$$\partial_{\xi}^{r}\partial_{x}^{q}K_l(t,x-z;\tau,\xi)=\partial_{\xi}^{r}\partial_{y}^{q}K_l(t,y;\tau,\xi)\Big|_{y=x-z}$$

then the expressions $\partial_\xi^r \partial_x^q K_l(t,x;\tau,\eta+\xi), \partial_\xi^r \partial_x^q K_l(t,x-z;\tau,\xi)$ and $\partial_\xi^r \partial_x^q K_l(t,x;\tau,\xi)$ are of the same type. Therefore, with regard for representation (14) and estimates (15), (16), (21), and (11), we have

$$\begin{aligned} \left|\partial_\xi^r \partial_x^q K_2(t,x;\tau,\xi)\right| &\le m2^{|r+q|_+}\Bigg(\sum_{|r_1|_+\le|r|_+} c_{r_1,q}c_{1,(r-r_1)}\int_\tau^{t_1}(t-\beta)^{-\frac{n+p_1+\gamma+|r_1+q|_+}{h}} \\ &\times(\beta-\tau)^{-\frac{n+p_1+\gamma}{h}}\int_{\mathbb{R}^n} e^{-\delta\left(\left(\frac{\|x-\eta-\xi\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}}+\left(\frac{\|\eta\|}{(\beta-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}\right)}d\eta d\beta+\sum_{|q_1|_+\le|q|_+}c_{q_1}c_{r,(q-q_1)} \\ &\times\int_{t_1}^{t}(\beta-\tau)^{-\frac{n+p_1+\gamma+|r+q-q_1|_+}{h}}(t-\beta)^{-\frac{n+p_1+\gamma}{h}}\int_{\mathbb{R}^n}e^{-\delta\left(\left(\frac{\|z\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}}+\left(\frac{\|x-z-\xi\|}{(\beta-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}\right)}dzd\beta\Bigg) \\ &\le m2^{|r+q|_+}c_\varepsilon e^{-\delta(1-\varepsilon)\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}(t-\tau)^{-\alpha n}\Bigg(\sum_{|r_1|_+\le|r|_+}c_{r_1,q}c_{1,(r-r_1)} \\ &\times\int_\tau^{t_1}(t-\beta)^{\alpha n-\frac{n+p_1+\gamma+|r_1+q|_+}{h}}(\beta-\tau)^{\alpha_0-1}d\beta+\sum_{|q_1|_+\le|q|_+}c_{q_1}c_{r,(q-q_1)} \\ &\times\int_{t_1}^{t}(t-\beta)^{\alpha_0-1}(\beta-\tau)^{\alpha n-\frac{n+p_1+\gamma+|r+q-q_1|_+}{h}}d\beta\Bigg), \quad |r|_+\le\alpha_*, |q|_+\le\alpha_*, \varepsilon\in(0;1). \end{aligned} \tag{23}$$

In view of the estimates

$$\int_\tau^{t_1}(t-\beta)^{\alpha n-\frac{n+p_1+\gamma+|r_1+q|_+}{h}}(\beta-\tau)^{\alpha_0-1}d\beta\le 2^{\frac{|r_1+q|_+}{h}}(t-\tau)^{2\alpha_0-\left(1+\frac{|r_1+q|_+}{h}\right)}B(\alpha_0,\alpha_0)$$

and

$$\int_{t_1}^{t}(t-\beta)^{\alpha_0-1}(\beta-\tau)^{\alpha n-\frac{n+p_1+\gamma+|r+q-q_1|_+}{h}}d\beta\le 2^{\frac{|r+q-q_1|_+}{h}}(t-\tau)^{2\alpha_0-\left(1+\frac{|r+q-q_1|_+}{h}\right)}B(\alpha_0,\alpha_0),$$

we get the inequality

$$\left|\partial_\xi^r \partial_x^q K_2(t,x;\tau,\xi)\right|\le c_{2,\xi}^{r,q}(t-\tau)^{2\alpha_0-\left(1+\alpha n+\frac{|r+q|_+}{h}\right)}B(\alpha_0,\alpha_0)e^{-\delta(1-\varepsilon)\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}.$$

By continuing stepwise the process of estimation, we obtain

$$\left|\partial_\xi^r \partial_x^q K_l(t,x;\tau,\xi)\right|\le c_{l,\varepsilon}^{r,q}(t-\tau)^{l\alpha_0-\left(1+\alpha n+\frac{|r+q|_+}{h}\right)},$$

$$\begin{aligned}\left|\partial_\xi^r \partial_x^q K_l(t,x;\tau,\xi)\right| &\le c_{l,\varepsilon}^{r,q}(t-\tau)^{l\alpha_0-\left(1+\alpha n+\frac{|r+q|_+}{h}\right)} \\ &\le e^{-\delta(1-(l-1)\varepsilon)\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}\Big(\prod_{j=1}^{l-1}B(\alpha_0,j\alpha_0)\Big),\end{aligned} \tag{24}$$

for all $|r|_+\le\alpha_*$, $|q|_+\le\alpha_*$, $\{x,\xi\}\subset\mathbb{R}^n$, $0\le\tau<t\le T$, $\varepsilon\in(0;1)$ and $l\in\mathbb{N}\backslash\{1\}$.

Let us pass to the estimation of the expression $\left|\partial_\xi^r \partial_x^q K_l(t,x;\tau,\xi)\right|$, which will be suitable for the establishment of the differentiability of the matrix function Φ with respect to the spatial variables. Directly from (24), we arrive at the existence of a number l^* such that

$$\left|\partial_\xi^r \partial_x^q K_{l^*}(t,x;\tau,\xi)\right| \le c_{l^*,\varepsilon}^{r,q} e^{-\delta(1-(l^*-1)\varepsilon)\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}} \Big(\prod_{j=1}^{l^*-1} B(\alpha_0, j\alpha_0) \Big).$$

Let us set $l_+ := \max\{l_*, l^*\}$, $l_- := \min\{l_*, l^*\}$, where l_* is the corresponding number from (22), $\varepsilon := \frac{1}{r_* l_+}$, $\delta_* := \delta\big(1 - \frac{1}{r_*}\big)$, $r_* > 2$, $T_0 := \max\{1, T\}$, and

$$c_*^0 := \max_{l \in \mathbb{N}_{l_+} \setminus \{1\}} \Big\{ c_{1,r}, c_{l,r}(\varepsilon) \Big(\prod_{j=1}^{l-1} B(\alpha_0, j\alpha_0) \Big), c_{r,q}, c_{l,\varepsilon}^{r,q} \Big(\prod_{j=1}^{l-1} B(\alpha_0, j\alpha_0) \Big) \Big\},$$

$c_* := c_*^0 (T_0)^{l_+ - l_-}$. Then (21) and (24) imply that, for all $\{x, \xi, \eta\} \subset \mathbb{R}^n$, $0 \le \tau < t \le T$, $|r|_+ \le \alpha_*$, and $|q|_+ \le \alpha_*$,

$$\left|\partial_\xi^r \partial_x^q K_{l_+}(t,x;\tau,\xi)\right| \le c_* e^{-\delta_* \left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}, \left|\partial_\xi^r K_{l_+}(t,\eta+\xi;\tau,\xi)\right| \le c_* e^{-\delta_* \left(\frac{\|\eta\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}.$$

In view of this result, estimate (10), the equality

$$\int_{\mathbb{R}^n} e^{-\delta_0 \left(\frac{\|x-y\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}}} \frac{dy}{(t-\beta)^{\alpha n}} = \int_{\mathbb{R}^n} e^{-\delta_0 \|z\|^{\frac{1}{1-\alpha}}} dz =: \widehat{E} < +\infty,$$

representation (14), and inequalities (15) and (16), we obtain

$$\begin{aligned} &\left|\partial_\xi^r K_{l_+ + 1}(t,\eta+\xi;\tau,\xi)\right| \\ &\quad \le \sum_{|r_1|_+ \le |r|_+} C_r^{r_1} \int_\tau^t d\beta \int_{\mathbb{R}^n} \left|\partial_\xi^{r_1} K_1(t,\eta+\xi;\beta,z+\xi) \partial_\xi^{r-r_1} K_{l_+}(\beta,z+\xi;\tau,\xi)\right| dz \\ &\quad \le m c_*^2 \Big(\sum_{|r_1|_+ \le |r|_+} C_r^{r_1} \Big) \int_\tau^t (t-\beta)^{\alpha_0 - 1} \int_{\mathbb{R}^n} e^{-\delta_* \left(\left(\frac{\|\eta-z\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\|z\|}{(\beta-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}} \right)} e^{-\frac{\delta}{r_*}\left(\frac{\|\eta-z\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}}} \frac{dz}{(t-\beta)^{n\alpha}} d\beta \\ &\quad \le m c_r^0 \widehat{E} c_*^2 B(\alpha_0, 1)(t-\tau)^{\alpha_0} e^{-\delta_* \left(\frac{\|\eta\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}, \qquad c_r^0 := \sum_{|r_1|_+ \le |r|_+} C_r^{r_1}; \end{aligned} \tag{25}$$

$$\begin{aligned}
&\left|\partial_\xi^r\partial_x^q K_{l_++1}(t,x;\tau,\xi)\right| \\
&\quad\le \sum_{|r_1|_+\le|r|_+} C_r^{r_1}\int_\tau^{t_1} d\beta\int_{\mathbb{R}^n}\left|\partial_\xi^{r_1}\partial_x^q K_1(t,x;\beta,\eta+\xi)\partial_\xi^{r-r_1}K_{l_+}(\beta,\eta+\xi;\tau,\xi)\right|d\eta \\
&\quad+\sum_{|q_1|_+\le|q|_+} C_q^{q_1}\int_{t_1}^{t} d\beta\int_{\mathbb{R}^n}\left|\partial_x^{q_1}K_1(t,x;\beta,x-z)\partial_\xi^r\partial_x^{q-q_1}K_{l_+}(\beta,x-z;\tau,\xi)\right|dz \\
&\quad\le mc_*^2\Bigg(\sum_{|r_1|_+\le|r|_+} C_r^{r_1}\int_\tau^{t_1}(t-\beta)^{\alpha_0-\left(1+\frac{|r_1+q|_+}{h}\right)} \\
&\qquad\times\int_{\mathbb{R}^n} e^{-\delta_*\left(\left(\frac{\|x-\eta-\xi\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}}+\left(\frac{\|\eta\|}{(\beta-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}\right)} e^{-\frac{\delta}{r_*}\left(\frac{\|x-\eta-\xi\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}}}\frac{d\eta}{(t-\beta)^{\alpha n}}d\beta \\
&\quad+\sum_{|q_1|_+\le|q|_+} C_q^{q_1}\int_{t_1}^{t}(t-\beta)^{\alpha_0-1}\int_{\mathbb{R}^n} e^{-\delta_*\left(\left(\frac{\|z\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}}+\left(\frac{\|x-z-\xi\|}{(\beta-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}\right)} e^{-\frac{\delta}{r_*}\left(\frac{\|z\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}}}\frac{dz}{(t-\beta)^{\alpha n}}d\beta \\
&\quad\le mc_*^2\widehat{E}e^{-\delta_*\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}\int_\tau^t(t-\beta)^{\alpha_0-1}d\beta\left(\left(\sum_{|r_1|_+\le|r|_+} C_r^{r_1}(t-t_1)^{-\frac{|r_1+q|_+}{h}}\right)+c_q\right) \\
&\quad\le mc_*^2\widehat{E}e^{-\delta_*\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}(t-\tau)^{\alpha_0}B(\alpha_0,1)\left(\left(2^{\frac{|r+q|_+}{h}}\sum_{|r_1|_+\le|r|_+} C_r^{r_1}(t-\tau)^{-\frac{|r_1+q|_+}{h}}\right)+c_q\right) \\
&\quad\le mc_{r,q}^0c_*^2\widehat{E}(2T_0)^{\frac{|r+q|_+}{h}}B(\alpha_0,1)(t-\tau)^{\alpha_0-\frac{|r+q|_+}{h}}e^{-\delta_*\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}},\ c_{r,q}^0:=c_r+c_q.
\end{aligned} \tag{26}$$

Applying the method of induction, we can verify firstly the validity of the estimate

$$\begin{aligned}
&\left|\partial_\xi^r K_{l_++l}(t,\eta+\xi;\tau,\xi)\right| \\
&\qquad\le c_*(mc_r^0c_*\widehat{E}(t-\tau)^{\alpha_0})^l e^{-\delta_*\left(\frac{\|\eta\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}\left(\prod_{j=1}^{l-1}B(\alpha_0,1+j\alpha_0)\right),
\end{aligned} \tag{27}$$

and, hence, the estimate

$$\begin{aligned}
\left|\partial_\xi^r\partial_x^q K_{l_++l}(t,x;\tau,\xi)\right| &\le c_*\left(mc_{r,q}^0c_*\widehat{E}(2T_0)^{\frac{|r+q|_+}{h}}\right)^l(t-\tau)^{l\alpha_0-\frac{|r+q|_+}{h}} \\
&\quad\times e^{-\delta_*\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}\left(\prod_{j=1}^{l-1}B(\alpha_0,1+j\alpha_0)\right),
\end{aligned} \tag{28}$$

for $|r|_+\le\alpha_*$, $|q|_+\le\alpha_*$, $(t,x;\tau,\xi)\in\Pi_T^2$ and $l\in\mathbb{N}\setminus\{1\}$.

The following propositions hold true.

Lemma 2.1. *The matrix function $\Phi(t,x;\tau,\xi)$ on the set Π_T^2 is a function differentiable with respect to each of the spatial variables x and ξ to the order α_* inclusively, and their derivatives satisfy the following estimates:*

$$\left|\partial_\xi^r\partial_x^q\Phi(t,x;\tau,\xi)\right|\le c_1(t-\tau)^{\alpha_0-(1+\alpha n+\frac{|r+q|_+}{h})}e^{-\delta_*\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}, \tag{29}$$

$$\left|\partial_\xi^r \Phi(t,\eta+\xi;\tau,\xi)\right| \leq c_2(t-\tau)^{\alpha_0-(1+\alpha n)} e^{-\delta_*\left(\frac{\|\eta\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}, \{\eta,\xi\} \subset \mathbb{R}^n \tag{30}$$

(here, the estimating constants c_1, c_2, and δ_* are independent of t, τ, x, ξ, η).

Proof. In any way, let us fix a point $(x_0;\xi_0)$ from $\mathbb{R}^{2n}$, and consider a ball $\mathbb{K}^\delta_{(x_0;\xi_0)}$ with radius $\delta > 0$, which is centered at the point $(x_0;\xi_0)$, in this space. Then, in view of structure (6) of the function Φ and the differentiability of the iterated kernels K_l with respect to spatial variables on $\mathbb{R}^{2n}$ to the order α_* inclusively, we can conclude that, in order to prove the differentiability of the matrix function Φ at the point $(x_0;\xi_0)$ to the indicated order, it is necessary only to prove the uniform convergence of the formally differentiated series (6) in the variables x and ξ on the set $\mathbb{K}^\delta_{(x_0;\xi_0)}$, $\delta > 0$ (at every fixed t and τ, $0 \leq \tau < t \leq T$):

$$\sum_{l=1}^{\infty} \partial_\xi^r \partial_x^q K_l(t,x;\tau,\xi), \quad |r|_+ \leq \alpha_*, \ |q|_+ \leq \alpha_*. \tag{31}$$

Directly from estimates (24) and (28) and the equality

$$\prod_{j=0}^{l-1} B(\alpha_0, 1+j\alpha_0) = \frac{\big(\Gamma(\alpha_0)\big)^l}{\Gamma(1+l\alpha_0)},$$

where $\Gamma(\cdot)$ is the Euler gamma-function, for $\{r,q\} \subset \mathbb{Z}^n_+$, $|r|_+ \leq \alpha_*$, $|q|_+ \leq \alpha_*$, and $(t,x;\tau,\xi) \in \Pi^2_T$, we have

$$\begin{aligned}
&\Big|\sum_{l=1}^{\infty} \partial_\xi^r \partial_x^q K_l(t,x;\tau,\xi)\Big| \leq \sum_{l=1}^{l_+} \Big|\partial_\xi^r \partial_x^q K_l(t,x;\tau,\xi)\Big| + \sum_{l=l_++1}^{\infty} \Big|\partial_\xi^r \partial_x^q K_l(t,x;\tau,\xi)\Big| \\
&\leq c_* \Bigg(\sum_{l=1}^{l_+} (t-\tau)^{l\alpha_0 - (1+\alpha n + \frac{|r+q|_+}{h})} + \sum_{l=1}^{\infty} \big(m c^0_{r,q} c_* \widehat{E}(2T_0)^{\frac{|r+q|_+}{h}}\big)^l (t-\tau)^{l\alpha_0 - \frac{|r+q|_+}{h}} \\
&\times \Big(\prod_{j=1}^{l-1} B(\alpha_0, 1+j\alpha_0)\Big)\Bigg) e^{-\delta_*\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}} \leq c_1(t-\tau)^{\alpha_0 - \left(1+\alpha n + \frac{|r+q|_+}{h}\right)} \times e^{-\delta_*\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}.
\end{aligned} \tag{32}$$

Frow here, we get the uniform convergence of series (31) in x and ξ and, hence, the validity of estimates (29).

Due to the corresponding estimates (21) and (27), we can verify analogously the validity of estimate (30). The lemma is proven. □

Lemma 2.2. *The volumetric potential $W(t,x;\tau,\xi)$ on the set Π^2_T is a function differentiable with respect to each of the spatial variables x and ξ to the orders $\alpha_* + p_1$ and α_* respectively inclusively. In this case,*

$$\begin{aligned}
\partial_\xi^r \partial_x^q W(t,x;\tau,\xi) &= \sum_{l=0}^{r} C_r^l \int_\tau^{t_1} d\beta \int_{\mathbb{R}^n} \partial_\xi^l \partial_x^q G(t,\beta;x-y-\xi) \partial_\xi^{r-l} \Phi(\beta, y+\xi;\tau,\xi) dy \\
&+ \int_{t_1}^{t} d\beta \int_{\mathbb{R}^n} \partial_x^q G(t,\beta;x-y) \partial_\xi^r \Phi(\beta,y;\tau,\xi) dy, \ |q|_+ \leq p_1, |r|_+ \leq \alpha_*,
\end{aligned} \tag{33}$$

$$\partial_\xi^r \partial_x^q W(t,x;\tau,\xi) = \sum_{l=0}^{r} C_r^l \int_\tau^{t_1} d\beta \int_{\mathbb{R}^n} \partial_\xi^l \partial_x^q G(t,\beta;x-y-\xi) \partial_\xi^{r-l} \Phi(\beta, y+\xi;\tau,\xi) dy$$
$$+ \int_{t_1}^{t} d\beta \int_{\mathbb{R}^n} \partial_\eta^k G(t,\beta;\eta) \partial_\xi^r \partial_x^{q-k} \Phi(\beta, x-\eta;\tau,\xi) d\eta, \ |r|_+ \le \alpha_*, \tag{34}$$
$$|k|_+ = p_1,\ p_1 < |q|_+ \le \alpha_* + p_1.$$

Proof. For $|q|_+ \le p_1$ and $|r|_+ \le \alpha_*$, we use the representation

$$W(t,x;\tau,\xi) = \int_\tau^{t_1} d\beta \int_{\mathbb{R}^n} G(t,\beta;x-y-\xi)\Phi(\beta,y+\xi;\tau,\xi)dy$$
$$+ \int_{t_1}^{t} d\beta \int_{\mathbb{R}^n} G(t,\beta;x-y)\Phi(\beta,y;\tau,\xi)dy.$$

From here, by the formal differentiation under the sign of integral, we obtain equality (33). Hence, in order to substantiate the validity of equality (33), it is sufficient to prove the uniform convergence of the following integrals in the variables x and ξ on $\mathbb{R}^{2n}$:

$$\mathcal{I}_1^{r,l,q}(t_1,x;\tau,\xi) := \int_\tau^{t_1} d\beta \int_{\mathbb{R}^n} |\partial_\xi^l \partial_x^q G(t,\beta;x-y-\xi)||\partial_\xi^{r-l}\Phi(\beta,y+\xi;\tau,\xi)|dy, \ |l|_+ \le |r|_+;$$
$$\mathcal{I}_2^{r,q}(t,x;t_1,\xi) := \int_{t_1}^{t} d\beta \int_{\mathbb{R}^n} |\partial_x^q G(t,\beta;x-y)||\partial_\xi^r \Phi(\beta,y;\tau,\xi)|dy. \tag{35}$$

This convergence becomes obvious, if we take condition (A) and the following estimates into account for $\{x,\xi\} \subset \mathbb{R}^n$ and $0 \le \tau < t \le T$:

$$\mathcal{I}_1^{r,l,q}(t_1,x;\tau,\xi) \le cc_2 \widehat{E} e^{-\delta_*\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}} (t-t_1)^{-\frac{n+\gamma+|l+q|_+}{h}}$$
$$\times \int_\tau^{t_1} (\beta-\tau)^{\alpha_0-1} d\beta, \ |l|_+ \le |r|_+; \tag{36}$$

$$\mathcal{I}_2^{r,q}(t,x;t_1,\xi) \le cc_1 \widehat{E} e^{-\delta_*\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}} (t_1-\tau)^{-\frac{n+p_1+\gamma+|r|_+}{h}} \int_{t_1}^{t} (t-\beta)^{\alpha_0-1+\frac{p_1-|q|_+}{h}} d\beta. \tag{37}$$

These estimates follow directly from (3), (29), and (30).

We now prove the validity of formula (34). For this purpose, we fix any $k \in \mathbb{Z}_+^n$ such that $|k|_+ = p_1$. Then, according to (33) for $p_1 < |q|_+ \le \alpha_* + p_1$ and $|r|_+ \le \alpha_*$, we have

$$\partial_\xi^r \partial_x^q W(t,x;\tau,\xi) = \sum_{l=0}^{r} C_r^l \partial_x^{q-k} \int_\tau^{t_1} d\beta \int_{\mathbb{R}^n} \partial_\xi^l \partial_x^k G(t,\beta;x-y-\xi) \partial_\xi^{r-l}\Phi(\beta,y+\xi;\tau,\xi)dy$$
$$+ \partial_x^{q-k} \int_{t_1}^{t} d\beta \int_{\mathbb{R}^n} \partial_\eta^k G(t,\beta;\eta) \partial_\xi^r \Phi(\beta,x-\eta;\tau,\xi)d\eta, \ (t,x;\tau,\xi) \in \Pi_T^2.$$

Hence, it remains to substantiate the possibility to introduce the operation ∂_x^{q-k} under the signs of the corresponding integrals. In other words, we should prove the uniform convergence in x and ξ of the following integrals on $\mathbb{R}^{2n}$ for $0 \le \tau < t \le T$:

$$\int_{\tau}^{t_1} d\beta \int_{\mathbb{R}^n} \partial_\xi^l \partial_x^q G(t,\beta;x-y-\xi)\Phi(\beta,y+\xi;\tau,\xi)dy,$$

$$\int_{t_1}^{t} d\beta \int_{\mathbb{R}^n} \partial_\eta^k G(t,\beta;\eta)\partial_\xi^r \partial_x^{q-k}\Phi(\beta,x-\eta;\tau,\xi)d\eta.$$

By reasoning similarly to the case of integrals (35) and using estimates (3), (29), and (30), we get the necessary convergence of the indicated integrals. The lemma is proven. □

The main result can be formulated as the following proposition.

Theorem 1. *Let the system* (1) *satisfy conditions (A) and (B). Then the corresponding function* $Z(t,x;\tau,\xi)$ *defined by equality* (4) *is a function differentiable with respect to each of the spatial variables* x *and* ξ *on the set* Π_T^2 *to the orders* $\alpha_* + p_1$ *and* α_* *respectively inclusively, and exists* $\delta > 0$ *for all* $\{r,q\} \subset \mathbb{Z}_+^n$, $|q|_+ \le \alpha_* + p_1$, $|r|_+ \le \alpha_*$, *exists* $c > 0$ *for all* $(t,x;\tau,\xi) \in \Pi_T^2$:

$$|\partial_\xi^r \partial_x^q Z(t,x;\tau,\xi)| \le c(t-\tau)^{-\frac{n+|r+q|_+ +\gamma}{h}} e^{-\delta\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}; \tag{38}$$

$$|\partial_\xi^k Z(t,x+\xi;\tau,\xi)| \le c_k(t-\tau)^{\beta_k - \frac{n+\gamma}{h}} e^{-\delta_1\left(\frac{\|x\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}, \tag{39}$$

where $|k|_+ \le \alpha_*$, $0 \le \tau < t \le T$, $\{x,\xi\} \subset \mathbb{R}^n$, $\beta_k := \begin{cases} 0, & k=0, \\ \alpha_0, & k \ne 0 \end{cases}$ *(here, the estimating constants are independent of* t, τ, x, *and* ξ*).*

Proof. With regard for structure (4) and the infinite differentiability of the function $G(t,\tau;\xi)$ with respect to the variable ξ, the smoothness of the function $Z(t,x;\tau,\xi)$ in the variables x and ξ becomes obvious directly from the assertion of Lemma 2.

Let $|q|_+ \le p_1$ and $|r|_+ \le \alpha_*$. Then, according to (33), we get

$$|\partial_\xi^r \partial_x^q Z(t,x;\tau,\xi)| \le |\partial_{x-\xi}^{r+q} G(t,\tau;x-\xi)| + \sum_{l=0}^{r} C_r^l \mathcal{I}_1^{r,l,q}(t_1,x;\tau,\xi) + \mathcal{I}_2^{r,q}(t,x;t_1,\xi).$$

From here, by using estimates (3), (36), and (37), we obtain assertion (38).

In a similar way, by using formula (34), we verify the validity of assertion (38) also for $p_1 < |q|_+ \le \alpha_*$ and $|r|_+ \le \alpha_*$.

Then, according to estimates (3) and (30), we have

$$\begin{aligned} Y_k(t,x;\tau,\xi) &:= \Big| \int_\tau^t d\beta \int_{\mathbb{R}^n} G(t,\beta;x-\zeta)\partial_\xi^k \Phi(\beta,\zeta+\xi;\tau,\xi)d\zeta \Big| \\ &\le cc_2 \int_\tau^t (t-\beta)^{\alpha_0 + \frac{p_1}{h} - 1}(\beta-\tau)^{\alpha_0 - 1} \int_{\mathbb{R}^n} \exp\Big\{ -\delta_0 \Big\{ \Big(\frac{\|x-\zeta\|}{(t-\beta)^\alpha}\Big)^{\frac{1}{1-\alpha}} \\ &\quad + \Big(\frac{\|\zeta\|}{(\beta-\tau)^\alpha}\Big)^{\frac{1}{1-\alpha}} \Big\}\Big\} \frac{dyd\beta}{((t-\beta)(\beta-\tau)^{\alpha n}}, \end{aligned}$$

$\delta_0 := \min\{\delta, \delta_*\}$, $|k|_+ \le \alpha_*$.

Using estimate (11) and equality (19), we get

$$Y_k(t,x;\tau,\xi) \leq c_\varepsilon (t-\tau)^{\alpha_0 - \frac{n+\gamma}{h}} e^{-\delta_0(1-\varepsilon)\left(\frac{\|x\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}, \quad \varepsilon \in (0;1),$$

where $|k|_+ \leq \alpha_*$, $0 \leq \tau < t \leq T$ and $\{x,\xi\} \subset \mathbb{R}^n$. From whence, with regard for inequality (3) and the representation

$$Z(t,x+\xi;\tau,\xi) = G(t,\tau;x) + \int_\tau^t d\beta \int_{\mathbb{R}^n} G(t,\beta;x-\zeta)\Phi(\beta,\zeta+\xi;\beta,\xi)d\zeta,$$

we arrive at estimate (39).

The theorem is proven. □

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На відміну від параболічних за Петровським систем, параболічні за Шиловим системи, взагалі кажучи, є параболічно нестійкими до зміни своїх коефіцієнтів. Саме тому сучасна теорія задачі Коші для систем класу Шилова розвинена на рівні систем із сталими, або залежними лише від часу t коефіцієнтами. Проблема побудови теорії задачі Коші для таких систем із змінними коефіцієнтами досі залишається відкритою. У даній роботі розглянуто новий клас лінійних параболічних систем рівнянь із частинними похідними першого порядку за t із змінними коефіцієнтами, який повністю охоплює клас Шилова систем з коефіцієнтами, залежними від t та невід'ємним родом. Головна частина диференціального виразу стосовно просторової змінної x кожної такої системи є параболічним за Шиловим виразом, коефіцієнти якого залежать від t тоді, як коефіцієнти групи молодших членів можуть залежати ще й від просторової змінної. Методом послідовного наближення побудовано фундаментальний розв'язок задачі Коші для систем із цього класу. З' ясовано умови мінімальної гладкості на коефіцієнти системи за змінною x, за яких існує фундаментальний розв'язок, досліджено його гладкість та одержано оцінки похідних цього розв'язку. Зазначені результати є важливими, зокрема, для встановлення коректної розв'язності задачі Коші для таких систем у різних функціональних просторах, одержанні форм зображення розв'язку цієї задачі та дослідженні його властивостей.

Ключові слова і фрази: фундаментальна матриця розв'язків, задача Коші, параболічні системи типу Шилова.