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T. V. Vasylyshyn, A. V. Zagorodnyuk

A QUESTION ON THE SPECTRA OF ALGEBRAS OF SYMMETRIC FUNCTIONS ON L_∞ RELATED TO THE MOMENT PROBLEM

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We consider a question on description of the set of characters of the algebra of bounded type symmetric analytic functions on $L_\infty[0,1]$ and establish some connection with the trigonometric moment problem.

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Рассматривается вопрос описания множества характеров алгебры симметрических аналитических функций ограниченного типа на $L_\infty[0,1]$ и установлена связь с тригонометрической проблемой моментов.

1. The problem. Let $H_{bs}(L_\infty)$ be the algebra of symmetric analytic functions of bounded type on the complex $L_\infty[0,1]$. Here "symmetric" means invariant with respect to measurable transformations of $[0,1]$ preserving the Lebesgue measure. It is known (see [4]) that the functions $R_n(x) = \int_{[0,1]}(x(t))^n\,dt$, $x(t) \in L_\infty[0,1]$ $(n \in \mathbb{N})$ form an algebraic basis in the algebra of all symmetric polynomials $\mathcal{P}_s(L_\infty) \subset H_{bs}(L_\infty)$ and that $\mathcal{P}_s(L_\infty)$ is a dense subspace of $H_{bs}(L_\infty)$. Hence, if ϕ is a complex homomorphism (i.e. a character) on $H_{bs}(L_\infty)$, then ϕ is completely defined by its values on R_n $(n \in \mathbb{N})$. Since point evaluation functionals $\delta_x(f) = f(x)$, $f \in H_{bs}(L_\infty)$, $x \in L_\infty[0,1]$ are characters, we have the following natural question.

Problem 1. *Describe the set of all sequences of complex numbers $\{a_n\}_{n=1}^\infty$ such that $R_n(x) = \int_{[0,1]}(x(t))^n\,dt = a_n$ for some $x \in L_\infty[0,1]$.*

2. Partial solutions and relations to the moment problem. In [5] it is proved that, for any finite sequence $\{a_n\}_{n=1}^m$ there is a function $x(\cdot) \in L_\infty[0,1]$ such that $R_n(x) = a_n$ $n \in \{1,2,\dots,m\}$ and $R_n(x) = 0$ for $n > m$.

Let us recall that a nondecreasing function $\sigma(\theta)$ is a solution of the trigonometric moment problem for a given sequence $\{c_k\}_{k=-\infty}^\infty$, $c_{-k} = \overline{c_k}$, $c_0 \in \mathbb{R}$ if

$$c_k = \frac{1}{2\pi}\int_{[-\pi,\pi]} \exp(ik\theta)\,d\sigma(\theta), \quad k \in \mathbb{Z}.$$

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It is known (see e.g. [1, Theorem 5.1.2]) that the trigonometric moment problem has a solution if and only if the Hermitian forms

$$\omega_n(\xi) = \sum_{\alpha,\beta=0}^{n} c_{\alpha-\beta}\xi_\alpha\overline{\xi_\beta},$$

$\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$ are nonnegative functions defined for all $n \in \mathbb{N}$. Making some simple calculations we can get the following result.

Theorem 1. *Let $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$. If $\sigma(\theta)$ is a strictly monotone solution of the trigonometric moment problem for $\{c_n\}_{n=-\infty}^{\infty}$, $c_n = a_n$ for $n > 0$, $c_0 = 1$ and $c_{-n} = \overline{a_n}$, then $x_\sigma = \exp(i\sigma^{-1}(2\pi t - \pi))$ satisfies $R_n(x_\sigma) = a_n$ $(n \in \mathbb{N})$.*

Corollary 1. *If $\{b_n\}_{n=1}^{\infty} \subset \mathbb{C}$ is such that for a given fixed number $z \in \mathbb{C}$, $\{a_n\}_{n=1}^{\infty} = \{b_n/z^n\}_{n=1}^{\infty}$ satisfies the condition of Theorem 1, then $x(t) = z\exp(i\sigma^{-1}(2\pi t - \pi))$ satisfies $R_n(x) = b_n$ $(n \in \mathbb{N})$.*

It is not difficult to construct a function $x(t) \in L_\infty[0, 1]$ such that $\{b_n\}_{n=1}^{\infty} = \{R_n(x)\}_{n=1}^{\infty}$ does not satisfy the conditions of Corollary 1.

A discrete analogue of Problem 1, namely, the problem to describe all sequences $\{a_n\}_{n=1}^{\infty}$ such that $F_n(x) = a_n$, where $x = (x_1, \ldots, x_n, \ldots) \in \ell_1$ and $F_n(x) = \sum_{k=1}^{\infty} x_k^n$, was investigated in [2, 3] and is related to the problem of description of the set of all characters on the algebra $H_{bs}(\ell_1)$ of symmetric analytic functions of bounded type on ℓ_1. Note that the discrete case is quite different from the continuous one. In particular, in [2] it is proved that, if for some $m > 0$, $F_n(x) = 0$ for all $n > m$, then $x = 0$, while there is a character ψ on $H_{bs}(\ell_1)$ such that $\psi(F_1) = 1$ and $\psi(F_n) = 0$ for $n > 1$. From results of [3] it follows that for every $x \in \ell_1$, $\sum_{n=0}^{\infty} G_n(z)z^n$ is a function of exponential type with zeros at $z_n = -1/x_n$ for all $x_n \neq 0$, where $G_0(z) = 1$, $G_1(z) = F_1(z)$, $nG_n(z) = F_1(z)G_{n-1}(z) - F_2(z)G_{n-2}(z) + \ldots + (-1)^{n-1}F_n(z)$, $n > 1$. However, the problem about description of all characters on $H_{bs}(\ell_1)$ is still open.

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Vasyl Stefanyk Precarpathian National University
taras_vasylyshyn@mail.ru, andriyzag@yahoo.com

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