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A NOTE ON APPROXIMATION OF CONTINUOUS FUNCTIONS ON NORMED SPACES

Let X be a real separable normed space X admitting a separating polynomial. We prove that each continuous function from a subset A of X to a real Banach space can be uniformly approximated by restrictions to A of functions, which are analytic on open subsets of X. Also we prove that each continuous function to a complex Banach space from a complex separable normed space, admitting a separating $*$-polynomial, can be uniformly approximated by $*$-analytic functions.

Key words and phrases: normed space, continuous function, analytic function, $*$-analytic function, uniform approximation, separating polynomial.

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The first known result on uniform approximation of continuous functions was obtained by Weierstrass in 1885. Namely, he showed that any continuous real-valued function on a compact subset K of a finitely dimensional real Euclidean space X can be uniformly approximated by restrictions on K of polynomials on X. For a compact subset K of a finitely dimensional complex Euclidean space X holds a counterpart of Stone-Weierstrass' theorem, according to which any continuous complex-valued function on K can be approximated by elements of any algebra, containing restrictions on K of polynomials on X and their conjugated functions. A general direction of investigations is to try to extend these results to topological linear spaces. Most of the obtained results concern separable Banach spaces, although in the paper [4] the authors obtained partial positive results for separable Fréchet spaces. A negative result belongs to Nemirovskii and Semenov, who in [7] built a continuous real-valued function on the unit ball K of the real space ℓ_2, which cannot be uniformly approximated by restrictions onto K of polynomials on ℓ_2. This result showed that in order to uniformly approximate continuous functions on Banach spaces we need a bigger class of functions than polynomials. The following fundamental result was obtained by Kurzweil [3].

Theorem 1. *Let X be any separable real Banach space that admits a separating polynomial, G be any open subset of X, and F be any continuous map from G to any real Banach space Y. Then for any $\varepsilon > 0$ there exists an analytic map H from G to Y such that $\|F(x) - H(x)\| < \varepsilon$ for all $x \in G$.*

Separating polynomials were introduced in [3] and are considered in reviews [2] and [6]. In order to define them and to obtain a counterpart of Kurzweil's Theorem for a complex Banach space X, in paper [5] were introduced notions, which we adapt below for complex normed spaces X and Y.

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A map B_{km} from X^{k+m} to Y is a map of type (k,m) if $B_{km}(x_1,\dots,x_k,x_{k+1},\dots,x_{k+m})$ is a nonzero map, which is k-linear with respect to x_i, $1\le i\le k$, and m-antilinear with respect to x_{k+j}, $1\le j\le m$.

Definition 1. *A map $B_n : X^n \to Y$ is $*$-n-linear if*

$$B_n(x_1,\dots,x_k,x_{k+1},\dots,x_{k+m}) = \sum_{k+m=n} c_{km}B_{km}(x_1,\dots,x_k,x_{k+1},\dots,x_{k+m}),$$

where for each k and m such that $k+m=n$, B_{km} is a map of type (k,m) and c_{km} is either 0 or 1, and at least one of c_{km} is non-zero.

Definition 2. *A map $F_n : X \to Y$ is called an n-homogeneous $*$-polynomial if there exists a $*$-n-linear map $B_n : X^n \to Y$ such that $F_n(x) = B_n(x,\dots,x)$ for all $x\in X$. Remark that F_0 is a constant map.*

Definition 3. *A map $F : X \to Y$ is a $*$-polynomial of degree j, if*

$$F = \sum_{n=0}^{j} F_n,$$

where F_n is an n-homogeneous continuous $*$-polynomial for each n and $F_j \ne 0$.

Definition 4. *A map $H : X \to Y$ is $*$-analytic if every point $x\in X$ has a neighborhood V such that*

$$H(x) = \sum_{n=0}^{\infty} F_n(x),$$

where for each n we have that F_n is an n-homogeneous continuous $*$-polynomial and the series $\sum\limits_{n=0}^{\infty} F_n(x)$ converges in V uniformly with respect to the norm of the space Y.

Definition 5. *Let X be a complex (resp. real) normed space. A $*$-polynomial (resp. polynomial) $P : X \to \mathbb{C}$ (resp. to $\mathbb{R}$) is called a separating $*$-polynomial (resp. polynomial) if $P(0)=0$ and $\inf\limits_{\|x\|=1} P(x) > 0$.*

Denote by $\widetilde{\mathcal{H}}(X,Y)$ the normed space of $*$-analytic functions from X to Y.

Theorem 2 ([5])**.** *Let X be any separable complex Banach space that admits a separating $*$-polynomial, Y be any complex Banach space, and $F : X \to Y$ be any continuous map. Then for any $\varepsilon > 0$ there exists a map $H \in \widetilde{\mathcal{H}}(X,Y)$ such that $\|F(x) - H(x)\| < \varepsilon$ for all $x \in X$.*

The aim of the present paper is to generalize Theorems 1 and 2 to normed spaces. To this end we need the following technical result.

Lemma 1. *If a real normed space X admits a separating polynomial q then its completion $\widehat{X}$ admits a separating polynomial too.*

Proof. Let $q = \sum\limits_{i\in I} q_i$ be a sum of homogeneous polynomials q_i on the space X. For each $i \in I$ there exists a polylinear form $h_i : X^{n_i} \to \mathbb{R}$ such that $q_i(x) = h_i(x,\dots,x)$ for each $x\in X$. Since h_i is a Lipschitz function on X^{n_i}, by [1, Theorem 4.3.17], it admits a continuous extension

$\widehat{h}_i$ on the space $\widehat{X}^{n_i}$, which is polylinear by the polylinearity of h_i. The map $\widehat{q}_i : \widehat{X} \to \mathbb{R}$ defined as $\widehat{q}_i(x) = \widehat{h}_i(x, \dots, x)$ for each $x \in \widehat{X}$ is an extension of the map q_i. Then the map $\widehat{q} = \sum_{i \in I} \widehat{q}_i$ is a continuous polynomial extension of the map q onto the space X. It is easy to show that the unit sphere S of the space X is dense in the unit sphere $\widehat{S}$ of the space $\widehat{X}$. Therefore $\inf_{x \in \widehat{S}} \widehat{q}(x) = \inf_{x \in S} q(x) > 0$, so $\widehat{q}$ is a separating polynomial for the space $\widehat{X}$. □

Theorem 3. *Let X be a separable real normed space that admits a separating polynomial, Y be a real Banach space, $A \subset X$, $f : A \to Y$ be a continuous function, and $\varepsilon > 0$. Then there are an open set $A_\varepsilon \supset A$ of X and an analytic function $f_\varepsilon : A_\varepsilon \to Y$ such that $\|f(x) - f_\varepsilon(x)\| < \varepsilon$ for all $x \in A$.*

Proof. Let $\widehat{X}$ be a completion of X. We build a cover of the set A by open in $\widehat{X}$ sets as follows. For each point $x \in A$ pick its neighborhood $O(x)$ open in $\widehat{X}$ such that $\|f(x') - f(x)\| < \varepsilon/3$ for all $x' \in O(x) \cap A$.

Put $\widehat{A}_\varepsilon = \bigcup_{x \in A} O(x)$. The topological space $\widehat{A}_\varepsilon$ is metrizable, and therefore paracompact, [1, 5.1.3]. Therefore, by [1, 5.1.9] there is a locally finite partition $\{\varphi_s : s \in S\}$ of the unity, subordinated to the cover $\{O(x) : x \in A\}$.

Now we construct an auxiliary function $f'_\varepsilon : \widehat{A}_\varepsilon \to Y$. First, for each index $s \in S$ we define a real number a_s as follows. If $\operatorname{supp} \varphi_s \cap A \neq \varnothing$, then we pick an arbitrary point $x_s \in \operatorname{supp} \varphi_s \cap A$, and we put $a_s = f(x_s)$. Otherwise, we put $a_s = 0$. Finally, put $f'_\varepsilon = \sum_{s \in S} a_s \varphi_s$.

Let $x \in A$. Put $S_x = \{s \in S : x \in \operatorname{supp} \varphi_s\}$. Then $\sum_{s \in S_x} \varphi_s(x) = 1$. Let $s \in S_x$ be any index. Thus there is an element $x_0 \in A$ such that $x \in \operatorname{supp} \varphi_s \subset O(x_0)$. Hence $x_s \in O(x_0)$ and

$$\|f(x) - a_s\| = \|f(x) - f(x_s)\| \leqslant \|f(x) - f(x_0)\| + \|f(x_0) - f(x_s)\| < 2\varepsilon/3.$$

Then

$$\begin{aligned}
\|f(x) - f'_\varepsilon(x)\| &= \Big\|f(x) - \sum_{s \in S} a_s \varphi_s(x)\Big\| = \Big\|\sum_{s \in S} f(x)\varphi_s(x) - \sum_{s \in S} a_s \varphi_s(x)\Big\| \\
&= \Big\|\sum_{s \in S_x} f(x)\varphi_s(x) - \sum_{s \in S_x} a_s \varphi_s(x)\Big\| \leqslant \sum_{s \in S_x} \|f(x)\varphi_s(x) - a_s \varphi_s(x)\| \\
&= \sum_{s \in S_x} \|f(x) - a_s\| \, \varphi_s(x) < \sum_{s \in S_x} (2\varepsilon/3)\varphi_s(x) = 2\varepsilon/3.
\end{aligned}$$

The function f'_ε is continuous on $\widehat{A}_\varepsilon$ as a sum of a family of continuous functions with a locally finite family of supports.

By Lemma 1, the space $\widehat{X}$ admits a separating polynomial. Therefore the space X satisfies the conditions of Theorem 1, so there exists a function $\widehat{f}_\varepsilon$ analytic on $\widehat{A}_\varepsilon$ such that $\|\widehat{f}_\varepsilon(x) - f'_\varepsilon(x)\| < \varepsilon/3$ for all $x \in \widehat{A}_\varepsilon$. Then for all $x \in A$ we have

$$\|f(x) - \widehat{f}_\varepsilon(x)\| \leqslant \|f(x) - f'_\varepsilon(x)\| + \|f'_\varepsilon(x) - \widehat{f}_\varepsilon(x)\| < \varepsilon.$$

It remains to put $A_\varepsilon = \widehat{A}_\varepsilon \cap X$ and let f_ε be the restriction of the map $\widehat{f}_\varepsilon$ to the set A_ε. □

For a complex normed space X we denote by $\widetilde{X}$ itself, considered as a real normed space, and by $\mathcal{H}(\widetilde{X}, Y)$ the real normed space of analytic functions from $\widetilde{X}$ to a Banach space Y.

Theorem 4. *Let X be any separable complex normed space that admits a separating $*$-polynomial, Y be any complex Banach space, and $F : X \to Y$ be any continuous map. Then for any $\varepsilon > 0$ there exists a map $H \in \widetilde{\mathcal{H}}(X, Y)$ such that $\|F(x) - H(x)\| < \varepsilon$ for each $x \in X$.*

Proof. The proof is almost identical to the proof of Theorem 4 from [5] with the following modifications. Instead of the application of Kurzweil's Theorem we apply Theorem 3. Instead of [5, Lemma 2] we use the fact (proof of which is similar to that of [5, Lemma 2]) that the identity map from a complex normed space $\widetilde{\mathcal{H}}(X, Y)$ to the real normed space $\mathcal{H}(\widetilde{X}, Y)$ is an isomorphism of real normed spaces. □

REFERENCES

[1] Engelking R. General topology. Heldermann, Berlin, 1989.

[2] Gonzalo R., Jaramillo J.A. *Separating polynomials on Banach spaces*. Extracta Math. 1997, **12** (2) 145–164.

[3] Kurzweil J. *On approximation in real Banach spaces*. Studia Math. 1954, **14**, 214–231.

[4] Mytrofanov M.A., Ravsky A.V. *Approximation of continuous functions on Frechet spaces*. J. Math. Sci. (N.Y.) 2012, **185**, 792–799. doi:10.1007/s10958-012-0961-6 (translation of Mat. Metodi Fiz.-Mekh. Polya 2011, **54** (3), 33–40. (in Ukrainian))

[5] Mitrofanov M.A. *Approximation of continuous functions on complex Banach spaces*. Math. Notes 2009, **86** (4), 530–541. doi:10.1134/S0001434609090302 (translation of Mat. Zametki 2009, **86** (4), 557–570. doi:10.4213/mzm5161 (in Russian))

[6] Mytrofanov M.A. *Separating polynomials, uniform analytical and separating functions*. Carpathian Math. Publ. 2015, **7** (2), 197–208. doi:10.15330/cmp.7.2.197-208 (in Ukrainian)

[7] Nemirovskii A.S., Semenov S.M. *On polynomial approximation of functions on Hilbert space*. Math. USSR Sb. 1973, **21** (2), 255–277. doi:10.1070/SM1973v021n02ABEH002016 (translation of Math. Sb. 1973, **92** (134), **2** (10), 257–281. (in Russian))

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Митрофанов М.А., Равський О.В. *Про апроксисмацію неперервних функції в нормованих просторах* // Карпатські матем. публ. — 2020. — Т.12, №1. — С. 107–110.

Нехай X є дійсним сепарабельним нормованим простором, що допускає відокремлювальний поліном. Показано, що непервні функції з підмножини A в X в дійсний банахів простір можуть бути рівномірно наближені аналітичними на відкритих підмножинах X. Також показано, що неперервні функції у комплексний банахів простір з комплексного сепарабельного нормованого простору, що допускає відокремлювальний $*$-поліном, можуть бути рівномірно наближені $*$-аналітичними функціями.

Ключові слова і фрази: нормований простір, неперервна функція, аналітична функція, $*$-аналітична функція, рівномірна апроксимація, відокремлювальний поліном.