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# WIENER WEIGHTED ALGEBRA OF FUNCTIONS OF INFINITELY MANY VARIABLES

In this article we consider a weighted Wiener type Banach algebra of infinitely many variables. The main result is a description of the spectrum of this algebra.

*Key words and phrases:* weighted algebra, spectrum of an algebra.

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## INTRODUCTION

Algebras of functions with absolutely summing Fourier series are called usually as Wiener type algebras. In [1] I. Gohberg, S. Goldberg, M.A. Kaashoek have given a description of a Wiener algebra with weights consisting of functions of several complex variables. They, in particular, described the spectrum (the set of multiplicative linear functionals) of this algebra. In this paper we consider a Wiener weighted algebra $W(w)$ of functions of infinitely many variables. Our main result is Theorem 1, where we have described the spectrum of $W(w)$. Also we consider the algebra $W_+(w)$ which consists of analytic functions on a Cartesian product of balls of radii $\rho_{2,k_m}$. The spectrum of the algebra $W_+(w)$ may be identified with the Cartesian product of these balls. In the case when $w(k_m) = 1$ for $k_m \geq 0$ these results was obtained by A.V. Zagorodnyuk and M.A. Mitrofanov in [2]. Spectra of algebras of analytic functions on Banach spaces were investigated by many authors in [3, 4, 5, 6]. For more informations on analytic functions on Banach spaces we refer the reader to [7, 8].

## 1 MAIN RESULTS

Let $c_{00}(\mathbb{Z})$ be a set of finite integer valued sequences $k = (k_\alpha)_{\alpha\in\mathbb{N}} = (k_1, \dots, k_l, 0, \dots)$, $k_\alpha \in \mathbb{Z}$ for all $\alpha \in \mathbb{N}, |k| = \sum_\alpha |k_\alpha|$. A *weight* is a map $w : c_{00}(\mathbb{Z}) \to [1;\infty)$ satisfying $w(k+s) \leqslant w(k)w(s)$, where $w(k) = w(k_1, \dots, k_l, 0, \dots)$, $w(s) = w(s_1, \dots, s_r, 0, \dots)$, $k+s = (k_1+s_1, \dots, k_n+s_n, 0, \dots)$. Let $W_0(w)$ be the space of all complex valued functions $f : l_\infty \to \mathbb{C}$ of the form $f(x) = \sum\limits_{|k|=0}^{m} a_k e^{i(k,x)} := \sum\limits_{|k|=0}^{m} a_{k_1,\dots,k_l} e^{i\sum_\alpha k_\alpha x_\alpha}$, $m \in \mathbb{N}$, with the norm

$$\|f\| := \sum_{|k|=0}^{m} |a_k| w(k) = \sum_{|k|=0}^{m} |a_{k_1,\dots,k_l}| w(k_1, \dots, k_l, 0, \dots), \tag{1}$$

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where $i$ is the imaginary unit. Let us denote by $W(w)$ the completion of $W_0(w)$ with respect to the norm (1). Hence every element $f$ on $W(w)$ has the form

$$f(x) = \sum_{|k|=0}^{\infty} a_k e^{i(k,x)} \tag{2}$$

and

$$\|f\| = \sum_{|k|=0}^{\infty} |a_k| w(k) < \infty. \tag{3}$$

**Lemma 1.** *Elements of the form* (2) *under condition* (3) *generate a weighted Banach-Wiener algebra.*

*Proof.* It is easy to see that $W(w)$ is an algebra. Let $f_n = \sum\limits_{|k_1|+\ldots+|k_l|=n} a_{k_1,\ldots,k_l} e^{i\sum\limits_{\alpha} k_\alpha x_\alpha}$, then $\|f_n\| = \sum\limits_{|k_1|+\ldots+|k_l|=n} |a_{k_1,\ldots,k_l}| w(k_1,\ldots,k_l)$. If $f, g \in W(w)$, then

$$\begin{aligned} \|f_n g_m\| &= \sum_{|k|+|s|=n+m} |a_{k_1,\ldots,k_l} b_{s_1,\ldots,s_r}| w(k_1,\ldots,k_l,s_1,\ldots,s_r) \\ &\leqslant \sum_{|k|+|s|=n+m} |a_k||b_s| w(k) w(s) = \sum_{|k|=n} |a_k| w(k) \sum_{|s|=m} |b_s| w(s) = \|f_n\| \|g_m\|. \end{aligned}$$

Thus $\|fg\| = \sum\limits_{n+m=0}^{\infty} \|f_n g_m\| \leqslant \sum\limits_{n+m=0}^{\infty} \|f_n\| \|g_m\| \leqslant \sum\limits_{n=0}^{\infty} \|f_n\| \sum\limits_{m=0}^{\infty} \|g_m\| = \|f\| \|g\|$.

The space $W(w)$ may be identified with the weighted $\ell_{1,w}(\mathbb{Z}^n)$ space of all sequences $\{a_{k_1,\ldots,k_l} w(k_1,\ldots,k_l)\}_{-\infty}^{\infty}$ which are in $\ell_1(\mathbb{Z}^n)$ and hence $W(w)$ is a Banach space. □

We describe multiplicative linear functionals on $W(w)$. Let $w(k_m) = w(0,\ldots,k_m,0,\ldots)$,

$$\rho_{1,k_m} := \sup_{k_m<0} \sqrt[k_m]{w(k_m)} = \lim_{k_m\to-\infty} \sqrt[k_m]{w(k_m)}, \tag{4}$$

$$\rho_{2,k_m} := \inf_{k_m>0} \sqrt[k_m]{w(k_m)} = \lim_{k_m\to\infty} \sqrt[k_m]{w(k_m)}. \tag{5}$$

Then $0 < \rho_{1,k_m} \leqslant \rho_{2,k_m} < \infty$.

For each $\lambda_\alpha, \alpha \in \mathbb{N}$, in the annulus $\rho_{1,k_\alpha} \leqslant |\lambda_\alpha| \leqslant \rho_{2,k_\alpha}$ we define functionals $h_\lambda(f)$ on $W(w)$ as follows

$$h_\lambda(f) = \sum_{|k|=0}^{\infty} a_{k_1,\ldots,k_l} \prod_{\alpha} \lambda_\alpha^{k_\alpha},$$

where $\lambda = (\lambda_1,\ldots,\lambda_l,0,\ldots)$. Then (4) and (5) imply that $h_\lambda(f)$ is well-defined. It is easy to see that $h_\lambda(f)$ is multiplicative and linear functional.

**Theorem 1.** *Each multiplicative linear functional $\varphi$ on $W(w)$ is an $h_\lambda(f)$ for some $\lambda_\alpha$ in $\rho_{1,\alpha} \leqslant |\lambda_\alpha| \leqslant \rho_{2,\alpha}$.*

*Proof.* Let $y_m \in W(w)$ be given by $y_m = e^{ix_m}$. The element $y_m$ is invertible in $W(w)$. For any positive integer $k_m$, we have $\|y_m^{k_m}\| = w(k_m)$ and $\|y_m^{-k_m}\| = w(-k_m)$. Since $\varphi(y_m^{-k_m}) = (\varphi(y_m))^{-k_m}$, it follows that $w(-k_m)^{-1/k_m} \leqslant |\varphi(y_m)| \leqslant w(k_m)^{1/k_m}$. But then (4) and (5) imply that $\lambda_m := \varphi(y_m)$ belongs to the annulus $\rho_{1,k_m} \leqslant |\lambda_m| \leqslant \rho_{2,k_m}$. Finally, observe that $f = \sum\limits_{|k|=0}^{\infty} a_{k_1,\ldots,k_l} e^{i\sum_\alpha k_\alpha x_\alpha} \in W(w)$ can be written as $f = \sum\limits_{|k|=0}^{\infty} a_{k_1,\ldots,k_l} \prod\limits_{\alpha} e^{ik_\alpha x_\alpha} = \sum\limits_{|k|=0}^{\infty} a_{k_1,\ldots,k_l} \prod\limits_{\alpha} y_\alpha^{k_\alpha}$

and the series converges in the norm of $W(w)$. Since $\varphi$ is continuous, linear and multiplicative functional, we conclude that

$$\varphi(f(x)) = \varphi\Big(\sum_{|k|=0}^{\infty} a_{k_1,\ldots,k_l}\prod_{\alpha} y_\alpha^{k_\alpha}\Big) = \sum_{|k|=0}^{\infty} a_{k_1,\ldots,k_l}\prod_{\alpha}\varphi(y_\alpha)^{k_\alpha} = \sum_{|k|=0}^{\infty} a_{k_1,\ldots,k_l}\prod_{\alpha}\lambda_\alpha^{k_\alpha} = h_\lambda(f).$$

□

On other words the spectrum of $W(w)$ may be identified with the set of point evaluation functionals at points $\{(x_1, x_2, \ldots, x_\alpha, \ldots) \in \ell_\infty : \rho_{1,\alpha} \leqslant |x_\alpha| \leqslant \rho_{2,\alpha}\}$.

**Remark 1.** If we consider the case $w(k_m) = 0$ for $k_m < 0$ , then the norm, which has been defined in (3), is actually a seminorm. Taking the quotient algebra with respect to the kernel of this seminorm we will obtain the algebra $W_+(w)$ which consists of analytic functions on a Cartesian product of balls of radii $\rho_{2,k_m}$. By the same way like in Theorem 1 we can show that the spectrum of $W_+(w)$ coincides with the set of point evaluation functionals at points $\{(x_1, x_2, \ldots, x_\alpha, \ldots) \in \ell_\infty : |x_\alpha| \leqslant \rho_{2,\alpha}\}$.

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Розглянуто зважену банахову алгебру Вінера від нескінченного числа змінних. Основним результатом є опис спектру цієї алгебри.

*Ключові слова і фрази:* зважена алгебра, спектр алгебри.