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SOME PROPERTIES OF BRANCHED CONTINUED FRACTIONS OF SPECIAL FORM

The fact that the values of the approximates of the positive definite branched continued fraction of special form are all in a certain circle is established for the certain conditions. The uniform convergence of branched continued fraction of special form, which is a particular case of the mentioned fraction, in the some limited parabolic region is investigated.

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INTRODUCTION

Several works are devoted to the establishment of different properties of branched continued fractions (BCF) of special form. For example, [1] is dedicated to the investigation of BCF with real positive and complex elements, [4] — to 1-periodic BCF of special form, [2] — to functional BCF with nonequivalent variables and BCF of special form with complex variables, [6] — to positive definite BCF of special form.

In this paper, using a representation of the approximants of BCF of special form (defined in [6]) through composition one- and two-dimensional fractional-linear maps, we have established that under certain conditions the values of the approximants of the positive definite BCF of special form

$$\Phi_0 + \cfrac{1}{b_{01} + z_{01} - \Phi_1 + \displaystyle\mathop{\mathrm{D}}_{s=2}^{\infty} \cfrac{-a_{0s}^2}{b_{0s} + z_{0s} - \Phi_s}}, \quad \Phi_p = \cfrac{1}{b_{1p} + z_{1p} + \displaystyle\mathop{\mathrm{D}}_{r=2}^{\infty} \cfrac{-a_{rp}^2}{b_{rp} + z_{rp}}}, \quad p \geq 0, \quad (1)$$

where a_{rs}, $r \geq 0$, $s \geq 0$, $r \neq 1$, $r + s \geq 2$, b_{rs}, $r \geq 0$, $s \geq 0$, $r + s \geq 1$, are complex numbers, z_{rs}, $r \geq 0$, $s \geq 0$, $r + s \geq 1$, are complex variables, are in a certain circle. Moreover, we investigated the converges uniformly of the BCF which is a particular case of the positive definite BCF of special form in the some limited parabolic domain.

1 PROPERTIES OF BCF OF SPECIAL FORM

We show that under certain conditions the values of the approximants of the positive definite BCF of special form (1) are in a certain circle. For this we prove the following lemma.

Lemma 1. *Let*

$$t_{r,s-1}(w_{r+1,s-1}) = b_{r,s-1} + z_{r,s-1} - \frac{a_{r+1,s-1}^2}{w_{r+1,s-1}}, \quad t_{0s}(w_{1s}, w_{0,s+1}) = b_{0s} + z_{0s} - \frac{1}{w_{1s}} - \frac{a_{0,s+1}^2}{w_{0,s+1}},$$

where $r \geq 1$, $s \geq 1$, and let

$$\begin{aligned}
y_{01} &= \operatorname{Im} z_{01} > 0, y_{1s} = \operatorname{Im} z_{1s} > 0, y_{rs} = \operatorname{Im} z_{rs} \geq 0, \quad r \geq 0, s \geq 0, r \neq 1, r+s \geq 2, \\
\beta_{rs} &= \operatorname{Im} b_{rs} \geq 0, \quad 0 \leq g_{rs} \leq 1, r \geq 0, s \geq 0, r+s \geq 1, \\
\alpha_{rs}^2 &= (\operatorname{Im} a_{rs})^2 \leq \beta_{rs}\beta_{r+\delta_{r0}-1,s-\delta_{r0}}(1 - g_{r+\delta_{r0}-1,s-\delta_{r0}})g_{rs}, \quad r \geq 0, s \geq 0, r \neq 1, r+s \geq 2,
\end{aligned} \tag{2}$$

where δ_{pq} is the Kronecker's delta. If $\operatorname{Im} w_{1,s+1} \geq \beta_{1,s+1}g_{1,s+1}$, $\operatorname{Im} w_{r-\delta_{r0}+1,s+\delta_{r0}} \geq \beta_{r-\delta_{r0}+1,s+\delta_{r0}}g_{r-\delta_{r0}+1,s+\delta_{r0}}$, where $r \geq 0$, $s \geq 0$, $r+s \geq 1$, then

$$\operatorname{Im} t_{rs}(w_{r+1,s}) \geq \beta_{rs}g_{rs} + y_{rs}, \quad r \geq 1, s \geq 0, \tag{3}$$

$$\operatorname{Im} t_{0s}(w_{1s}, w_{0,s+1}) \geq \beta_{0s}g_{0s} + y_{0s}, \quad s \geq 1. \tag{4}$$

Proof. The validity of the inequalities (3) follows from [5, Lemma 17.1]. We show that the inequalities (4) are valid. It is obvious that for arbitrary s, $s \geq 1$, provided that $\beta_{1s}g_{1s} + y_{1s} > 0$ the image of half-plane $\operatorname{Im} w_{1s} \geq \beta_{1s}g_{1s} + y_{1s}$ under the transformation $t = w^{-1}$ is the circle

$$\left| \frac{1}{w_{1s}} + \frac{i}{2(\beta_{1s}g_{1s} + y_{1s})} \right| \leq \frac{1}{2(\beta_{1s}g_{1s} + y_{1s})}.$$

Hence $\operatorname{Im}(1/w_{1s}) \leq 0$. Let all $y_{0s} > 0$. By the lemma for arbitrary s, $s \geq 1$, we have

$$\operatorname{Im} w_{0,s+1} \geq \beta_{0,s+1}g_{0,s+1}\frac{\beta_{0s}(1 - g_{0s})}{\beta_{0s}(1 - g_{0s}) + y_{0s}} \geq \frac{\alpha_{0,s+1}^2}{\beta_{0s}(1 - g_{0s}) + y_{0s}}.$$

Therefore,

$$\left| w_{0,s+1} + \frac{ia_{0,s+1}^2}{2(\beta_{0s}(1 - g_{0s}) + y_{0s})} \right| \geq \frac{|a_{0,s+1}^2|}{2(\beta_{0s}(1 - g_{0s}) + y_{0s})}$$

or

$$\left| \frac{w_{0,s+1}}{a_{0,s+1}^2} + \frac{i}{2(\beta_{0s}(1 - g_{0s}) + y_{0s})} \right| \geq \frac{1}{2(\beta_{0s}(1 - g_{0s}) + y_{0s})}. \tag{5}$$

The image of (5) under the transformation $w = 1/z$ is the half-plane

$$\operatorname{Im} \frac{a_{0,s+1}^2}{w_{0,s+1}} \leq \beta_{0s}(1 - g_{0s}) + y_{0s}.$$

Next, for arbitrary s, $s \geq 1$, we have

$$\operatorname{Im} t_{0s}(w_{1s}, w_{0,s+1}) = \beta_{0s} + y_{0s} - \operatorname{Im} \frac{1}{w_{1s}} - \operatorname{Im} \frac{a_{0,s+1}^2}{w_{0,s+1}} \geq \beta_{0s} + y_{0s} - \operatorname{Im} \frac{a_{0,s+1}^2}{w_{0,s+1}} \geq \beta_{0s}g_{0s}.$$

Going to the limit in the last inequality for $y_{0s} \to 0$, we obtain $\beta_{0s} - \operatorname{Im} \frac{a_{0,s+1}^2}{w_{0,s+1}} \geq \beta_{0s}g_{0s}$. Thus, $\operatorname{Im} t_{0s}(w_{1s}, w_{0,s+1}) = \beta_{0s} + y_{0s} - \operatorname{Im} \frac{1}{w_{1s}} - \operatorname{Im} \frac{a_{0,s+1}^2}{w_{0,s+1}} \geq \beta_{0s}g_{0s} + y_{0s}$, which had to be proved. $\square$

Since the images of half-planes $\operatorname{Im} w_{10} \geq \beta_{10}g_{10} + y_{10}$ and $\operatorname{Im} w_{01} \geq \beta_{01}g_{01} + y_{01}$ under the transformation $t = t_0(w) = 1/w$ is respectively circles (nested or coincide)

$$\left| t + \frac{i}{2(\beta_{10}g_{10} + y_{10})} \right| \leq \frac{1}{2(\beta_{10}g_{10} + y_{10})}, \quad \left| t + \frac{i}{2(\beta_{01}g_{01} + y_{01})} \right| \leq \frac{1}{2(\beta_{01}g_{01} + y_{01})}$$

for $\beta_{10}g_{10} + y_{10} > 0$, $\beta_{01}g_{01} + y_{01} > 0$, then the image of transformation

$$t = t_0(w_{10}, w_{01}) = \frac{1}{w_{10}} + \frac{1}{w_{01}}$$

is the circle

$$\left| t + \frac{i(\beta_{10}g_{10} + y_{10} + \beta_{01}g_{01} + y_{01})}{2(\beta_{10}g_{10} + y_{10})(\beta_{01}g_{01} + y_{01})} \right| \leq \frac{\beta_{10}g_{10} + y_{10} + \beta_{01}g_{01} + y_{01}}{2(\beta_{10}g_{10} + y_{10})(\beta_{01}g_{01} + y_{01})},$$

which we denote by $K_0(\mathbf{z})$, where $\mathbf{z} = (z_{10}, z_{01}, z_{20}, z_{11}, z_{02}, \ldots)$ is infinite-dimensional vector.

For arbitrary n, $n \geq 1$, we define $K_n(\mathbf{z})$ as the map of the region

$$\operatorname{Im} w_{1n} \geq \beta_{1n}g_{1n}, \quad \operatorname{Im} w_{r-\delta_{r0}+1, s+\delta_{r0}} \geq \beta_{r-\delta_{r0}+1, s+\delta_{r0}} g_{r-\delta_{r0}+1, s+\delta_{r0}},$$

where $r \geq 0$, $s \geq 0$, $r + s = n$, under the transformation

$$T_n(w_{n+1,0}, w_{n1}, \ldots, w_{0,n+1}) = \Phi_0^n + \frac{1}{b_{01} + z_{01} - \Phi_1^{n-1}} - \frac{a_{02}^2}{b_{02} + z_{02} - \Phi_2^{n-2}}$$

$$- \frac{a_{03}^2}{b_{03} + z_{03} - \Phi_3^{n-3}} - \cdots - \frac{a_{0,n-1}^2}{b_{0,n-1} + z_{0,n-1} - \Phi_{n-1}^1} - \frac{a_{0n}^2}{b_{0n} + z_{0n} - \dfrac{1}{w_{1n}} - \dfrac{a_{0,n+1}^2}{w_{0,n+1}}},$$

where

$$\Phi_k^{n-k} = \frac{1}{b_{1k} + z_{1k}} - \frac{a_{2k}^2}{b_{2k} + z_{2k}} - \frac{a_{3k}^2}{b_{3k} + z_{3k}} - \cdots - \frac{a_{n-k,k}^2}{b_{n-k,k} + z_{n-k,k}} - \frac{a_{n-k+1,k}^2}{w_{n-k+1,k}}, \quad 0 \leq k \leq n - 1.$$

Applying lemma 1 and taking into account (3) and (4), we have

$$K_0(\mathbf{z}) \supseteq K_1(\mathbf{z}) \supseteq K_2(\mathbf{z}) \supseteq \ldots \tag{6}$$

Since (see [3, pp. 15–16]) $T_n(\underbrace{\infty, \infty, \ldots, \infty}_{n+2}) = f_n(\mathbf{z})$, where $f_n(\mathbf{z})$ is the nth approximant of the BCF (1), then $f_n(\mathbf{z}) \in K_n(\mathbf{z})$, $n \geq 1$. Hence we prove the following theorem.

Theorem 1. *If the conditions* (2) *holds, where*

$$\beta_{rs} \geq 0, \quad \beta_{1s}g_{1s} + y_{1s} > 0, \quad \beta_{01}g_{01} + y_{01} > 0, \quad y_{rs} \geq 0, \quad r \geq 0, \, s \geq 0, \, r + s \geq 1,$$

then the approximants $f_n(\mathbf{z})$, $n \geq 1$, *of the BCF* (1) *satisfy the inequalities*

$$\operatorname{Im} f_n(\mathbf{z}) \leq 0, \quad |f_n(\mathbf{z})| \leq \frac{\beta_{10}g_{10} + y_{10} + \beta_{01}g_{01} + y_{01}}{(\beta_{10}g_{10} + y_{10})(\beta_{01}g_{01} + y_{01})}, \quad n \geq 1.$$

In [6] the notion of the nth denominator $B_n(\mathbf{z})$ of the approximant $f_n(\mathbf{z})$, $n \geq 1$, of BCF (1) is given. By arguments similar to the proof of the [3, Theorem 4.8], we can show that following theorem holds.

Theorem 2. *If the conditions* (2) *holds, where*

$$\beta_{rs}g_{rs} > 0, \quad r \geq 0, \, s \geq 0, \, r + s \geq 1,$$

then the denominators $B_n(\mathbf{z})$, $n \geq 1$, *of the BCF* (1) *are different from zero for*

$$\operatorname{Im} z_{rs} \geq 0, \quad r \geq 0, \, s \geq 0, \, r + s \geq 1.$$

2 POSITIVE DEFINITE BCF OF SPECIAL FORM AND THE PARABOLA THEOREM

Putting $z_{rs} = 0$, $b_{rs} = i$, $r \geq 0$, $s \geq 0$, $r + s \geq 1$, in BCF (1), we obtain

$$\hat{\Phi}_0 + \cfrac{1}{i - \hat{\Phi}_1 - \underset{s=2}{\overset{\infty}{D}} \cfrac{a_{0s}^2}{i - \hat{\Phi}_s}}, \quad \hat{\Phi}_p = \cfrac{1}{i - \underset{r=2}{\overset{\infty}{D}} \cfrac{a_{rp}^2}{i}}, \quad p \geq 0. \tag{7}$$

Using the equivalent transformation [5, pp. 19–20], we put $\rho_0 = i, \rho_{rs} = i, r \geq 0, s \geq 0, r + s \geq 1$, and BCF (7) reduce to

$$-i\tilde{\Phi}_0 + \cfrac{-i}{1 + \tilde{\Phi}_1 + \underset{s=2}{\overset{\infty}{D}} \cfrac{a_{0s}^2}{1 + \tilde{\Phi}_s}}, \quad \tilde{\Phi}_p = \cfrac{1}{1 + \underset{r=2}{\overset{\infty}{D}} \cfrac{a_{rp}^2}{1}}, \quad p \geq 0. \tag{8}$$

Next, putting $\rho_0 = 1/(1 + \tilde{\Phi}_1)$, $\rho_{0s} = 1/(1 + \tilde{\Phi}_{s+1})$, $s \geq 1$, we reduce the fraction (8) to the fraction with partial denominators equal to unity

$$-i\tilde{\Phi}_0 + \cfrac{\cfrac{-i}{1 + \tilde{\Phi}_1}}{1 + \underset{s=2}{\overset{\infty}{D}} \cfrac{\cfrac{a_{0s}^2}{(1 + \tilde{\Phi}_{s-1})(1 + \tilde{\Phi}_s)}}{1}}. \tag{9}$$

Let

$$|a_{rs}^2| - \operatorname{Re} a_{rs}^2 \leq \frac{1}{2}, \quad |a_{rs}^2| \leq M, \quad M \geq 0, \quad r \geq 0, \; s \geq 0, \; r \neq 1, \; r + s \geq 2. \tag{10}$$

Then according to [5, Theorem 18.1] the continued fraction $\tilde{\Phi}_s$, $s \geq 0$, converges uniformly and according to [5, Theorem 14.3] the value of these fractions and of its approximants are in the domain $|z - 1| \leq 1, z \neq 0$.

We take an arbitrary $s, s \geq 1$. The fraction $1/(1 + \tilde{\Phi}_s)$ we write in the form $w = 1/(1 + z)$. Hence $z = (1 - w)/w$. Since $|z - 1| \leq 1, z \neq 0$, then

$$\left| \frac{1 - w}{w} - 1 \right| \leq 1, \; w \neq 1 \quad \text{or} \quad |1 - 2w| \leq |w|, \; w \neq 1.$$

Let $w = x + iy$. Then

$$|1 - 2x - i2y| \leq |x + iy|, \quad (1 - 2x)^2 + 4y^2 \leq x^2 + y^2, \quad 3x^2 - 4x + 1 + 3y^2 \leq 0,$$

$$\left(x - \frac{2}{3} \right)^2 + y^2 \leq \frac{1}{9}, \quad \left| w - \frac{2}{3} \right| \leq \frac{1}{3}.$$

Thus, the value of the fractions $1/(1 + \tilde{\Phi}_s)$, $s \geq 1$, and of its approximants are in the domain $|w - 2/3| \leq 1/3, w \neq 1$.

We put $1/(1 + \tilde{\Phi}_s) = r_s e^{i\varphi_s}$, $s \geq 1$. Since the line $y = kx$ touches to the circle $3x^2 - 4x + 1 + 3y^2 = 0$ for $k = \pm 1/\sqrt{3}$, then $-\pi/6 \leq \varphi_s \leq \pi/6, s \geq 1$.

The following inequalities are valid for all $s \geq 2$

$$\left| \frac{1}{(1 + \tilde{\Phi}_{s-1})(1 + \tilde{\Phi}_s)} \right| \leq \cos^2 \frac{\varphi_{s-1} + \varphi_s}{2}, \quad -\frac{\pi}{6} \leq \varphi_{s-1}, \varphi_s \leq \frac{\pi}{6}. \tag{11}$$

Indeed, let $x = r \cos \varphi$, $y = r \sin \varphi$. Then the circle equation $3x^2 - 4x + 1 + 3y^2 = 0$ in polar coordinates we write in the form $3r^2 - 4r \cos \varphi + 1 = 0$ or

$$r = \frac{2 \cos \varphi \pm \sqrt{4 \cos^2 \varphi - 3}}{3}, \quad -\frac{\pi}{6} \le \varphi \le \frac{\pi}{6}.$$

The inequalities (11) are equivalent to the inequalities

$$\frac{2 \cos \varphi_{s-1} \pm \sqrt{4 \cos^2 \varphi_{s-1} - 3}}{3} \frac{2 \cos \varphi_s \pm \sqrt{4 \cos^2 \varphi_s - 3}}{3} \le \cos^2 \frac{\varphi_{s-1} + \varphi_s}{2}, \qquad (12)$$

where $-\pi/6 \le \varphi_{s-1}, \varphi_s \le \pi/6$, $s \ge 2$.

To prove inequalities (12) is sufficient to show the validity of following inequalities

$$\frac{2 \cos \varphi_{s-1} + \sqrt{4 \cos^2 \varphi_{s-1} - 3}}{3} \frac{2 \cos \varphi_s + \sqrt{4 \cos^2 \varphi_s - 3}}{3} \le \cos^2 \frac{\varphi_{s-1} + \varphi_s}{2}, \qquad (13)$$

where $-\pi/6 \le \varphi_{s-1}, \varphi_s \le \pi/6$, $s \ge 2$.

Since $\sqrt{4 \cos^2 \varphi - 3} \le \cos \varphi$ for $-\pi/6 \le \varphi \le \pi/6$, then, estimating the top left side of the inequality (13), for any $-\pi/6 \le \varphi_{s-1}, \varphi_s \le \pi/6$, $s \ge 2$, we have

$$\cos \varphi_{s-1} \cos \varphi_s \le \cos^2 \frac{\varphi_{s-1} + \varphi_s}{2} \quad \text{or} \quad \cos(\varphi_{s-1} - \varphi_s) \le 1.$$

That is, the inequalities (13) holds.

Applying relations (10) and (11), for any $s \ge 2$ we have

$$\left| \frac{a_{0s}^2}{(1 + \tilde{\Phi}_{s-1})(1 + \tilde{\Phi}_s)} \right| - \text{Re} \frac{a_{0s}^2 e^{-i(\varphi_{s-1} + \varphi_s)}}{(1 + \tilde{\Phi}_{s-1})(1 + \tilde{\Phi}_s)} \le \frac{1}{2} \cos^2 \frac{\varphi_{s-1} + \varphi_s}{2}, \quad -\frac{\pi}{6} \le \varphi_{s-1}, \varphi_s \le \frac{\pi}{6},$$

$$\left| \frac{a_{0s}^2}{(1 + \tilde{\Phi}_{s-1})(1 + \tilde{\Phi}_s)} \right| \le M, \quad M > 0.$$

According to [7, Theorem 4.40] the continued fraction

$$\cfrac{\cfrac{-i}{1 + \tilde{\Phi}_1}}{1 + \overset{\infty}{\underset{s=2}{D}} \cfrac{\cfrac{a_{0s}^2}{(1 + \tilde{\Phi}_{s-1})(1 + \tilde{\Phi}_s)}}{1}}$$

converges uniformly. Hence, the fraction (9) converges uniformly too. From equivalence fractions (7) and (9) we conclude that BCF (7) converges uniformly, if the conditions (10) holds.

Hence, we have, if we change the notation, the following theorem holds.

Theorem 3. *BCF*

$$\Phi_0 + \cfrac{1}{1 + \Phi_1 + \overset{\infty}{\underset{s=2}{D}} \cfrac{a_{0s}}{1 + \Phi_s}}, \quad \Phi_p = \cfrac{1}{1 + \overset{\infty}{\underset{r=2}{D}} \cfrac{a_{rp}}{1}}, \quad p \ge 0,$$

converges uniformly for all a_{rs} in the domain

$$P_M = \{z \in \mathbb{C} : \ |z| - \text{Re} \, z \le 1/2, \quad |z| < M\}$$

for every constant $M > 0$.

References

[1] Antomova T.M., Bodnar D.I. *Region convergence of branched continued fractions of special form.* Approx. Theor. and its Appl.: Pr. Inst. Matem. NAS Ukr. 2000, **31**, 19–32. (in Ukrainian)

[2] Baran O.E. *Some convergence regions of branched continued fractions of special form.* Carpathian Math. Publ. 2013, **5** (1), 4–13. doi: 10.15330/cmp.5.1.4-13 (in Ukrainian)

[3] Bodnar D.I. Branched continued fraction. Naukova Dumka, Kiev, 1986. (in Russian)

[4] Bodnar D.I., Bubnyak M.N. *On convergence of 1-periodic branched continued fraction of special form*. Math. Bull. Shevchenko Sci. Soc., **8**, 5–16. (in Ukrainian)

[5] Wall H.S. Analytic theory of continued fractions. Van Nostrand, New York, 1948.

[6] Dmytryshyn R.I. *Positive definite branched continued fractions of special form.* Carpathian Math. Publ. 2013, **5** (2), 225–230. doi: 10.15330/cmp.5.2.225-230

[7] Jones W.B., Thron W.J. Continued fractions: Analytic theory and applications. In: Encyclopedia of Mathematics and its Applications, 11. London, Amsterdam, Don Mills, Ontario, Sydney, Tokyo, 1980.

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Встановлено, що за певних умов значення підхідних дробів додатно визначеного гіллястого ланцюгового дробу спеціального вигляду належать деякому кругу та досліджено рівномірну збіжність гіллястого ланцюгового дробу спеціального вигляду, який є частинним випадком такого дробу, в деякій обмеженій параболічній області.

Ключові слова і фрази: гіллястий ланцюговий дріб спеціального вигляду.