ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2015, 7 (2), 148–154
doi:10.15330/cmp.7.2.148-154

http://www.journals.pu.if.ua/index.php/cmp
Карпатськi матем. публ. 2015, Т.7, №2, С.148–154

BODNAR D.I., BUBNIAK M.M.

ON CONVERGENCE OF $(2,1,\ldots,1)$-PERIODIC BRANCHED CONTINUED FRACTION OF THE SPECIAL FORM

$(2,1,\ldots,1)$-periodic branched continued fraction of the special form is defined. Conditions of convergence are established for 2-periodic continued fraction and $(2,1,\ldots,1)$-periodic branched continued fraction of the special form. Truncation error bounds are estimated for these fractions under additional conditions.

Key words and phrases: periodic branched continued fractions of the special form, convergence.

Ternopil National Economic University, 11 Lvivska str., 46020, Ternopil, Ukraine
E-mail: bodnar4755@ukr.net (Bodnar D.I.), maria.bubnyak@gmail.com (Bubniak M.M.)

INTRODUCTION

Periodic continued fractions are an important subclass of continued fractions

$$b_0 + \mathop{\mathrm{D}}_{k=1}^{\infty} \frac{a_k}{b_k} = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}} = b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \ldots, \tag{1}$$

where $a_i, b_0, b_i \in \mathbb{C}$; $i \geq 1$. A fraction (1) is called p-periodic, if its elements satisfy the following conditions: $a_{pn+k} = a_k$ and $b_{pn+k} = b_k$; $n \geq 0$; $1 \leq k \leq p$; $p \in \mathbb{N}$. L. Euler, D. Bernoulli, E. Kahl, E. Galios, A. Pringsheim, W. Leighton, O. Perron, R. Lane, H. Wall, W. Jones, W. Thron, H. Waadeland, L. Loretzen, A. F. Beardon etc. investigated p-periodic fractions. The reviews of corresponding results can be found in [5–7]. It is known (see [5, p. 181]), that the set

$$\Omega = \{z \in \mathbb{C} : |\arg(z + 1/4)| < \pi\} \tag{2}$$

is the convergence set of the 1-periodic continued fraction

$$1 + \frac{c|}{|1} + \frac{c|}{|1} + \ldots. \tag{3}$$

Moreover, attracting and repelling fixed points of the linear fractional transformation $t(\omega) = 1 + c/\omega$ are the points

$$x = (1 + \sqrt{1+4c})/2, \quad y = (1 - \sqrt{1+4c})/2. \tag{4}$$

In [5, p. 49] it is proved that the the following relations are valid for the fraction (3)

$$P_n = \frac{x^{n+2} - y^{n+2}}{x - y}, \quad Q_n = \frac{x^{n+1} - y^{n+1}}{x - y}, \quad n \geq 1, \quad \lim_{n\to\infty} \frac{P_n}{Q_n} = x. \tag{5}$$

УДК 515.12+512.58
2010 *Mathematics Subject Classification:* 18B30, 54B30.

1 Main Results

We consider the branched continued fraction (BCF) of the special form

$$1+\mathop{\mathrm{D}}_{k=1}^{\infty}\sum_{i_k=1}^{i_{k-1}}\frac{a_{i(k)}}{1}, \tag{6}$$

where $a_{i(k)}\in\mathbb{C}$, $i(k)\in\mathcal{I}$, $\mathcal{I}$ is a set of multiindex, $\mathcal{I}=\{i_1i_2\ldots i_k: 1\le i_k\le i_{k-1}; k\ge 1; i_0=N\}$, N is a fixed natural number. Some results according to these BCF are in [3,4].

Continued fraction

$$1+a_{i(m)}\left(1+\mathop{\mathrm{D}}_{q=1}^{\infty}\frac{a_{i(m+q)}}{1}\right)^{-1},$$

where $i(m)\in\mathcal{I}$, $i_m=i_{m+q}=r$, $q\ge 1$, is called the $i(m)$-th branch of the r-th order of BCF (6).

Definition. *A fraction (6) is called $\overrightarrow{p}$-periodic branched continued fraction of the special form, where $\overrightarrow{p}=(p_1,p_2,\ldots,p_N)$, $p_j\in\mathbb{N}$, $j=\overline{1,N}$, if all $i(m)$-th branches are the identical p_{i_m}-periodic continued fraction for each fixed i_m.*

Let BCF (6) be a $\overrightarrow{p}$-periodic fraction. Than its elements satisfy the following conditions

$$a_{\underbrace{r\ldots r}_{q}}=a_{\underbrace{r\ldots r}_{s}}\quad or\quad a_{i(m)\underbrace{r\ldots r}_{q}}=a_{\underbrace{r\ldots r}_{s}}, \tag{7}$$

where $q\ge 1$; $q=n\cdot p_r+s$; $r=\overline{1,N}$; $s=\overline{1,p_r}$; $m\ge 1$; $i(m)\in\mathcal{I}$; $r<i_m$; $n\ge 0$. Each $i(m)$-th branch of the r-th order is called the r-th branch of such fraction.

We introduce the notation $a_{\underbrace{r\ldots r}_{s}}=c_{r,s}$ for elements of the fraction (6). Then $\overrightarrow{p}$-periodic BCF can be written as follows

$$1+\mathop{\mathrm{D}}_{k=1}^{\infty}\sum_{i_k=1}^{i_{k-1}}\frac{c_{i_k,s}}{1}. \tag{8}$$

We investigate the convergence of $(2,1,\ldots,1)$-periodic BCF with N branches

$$1+\cfrac{c_{1,1}}{1+\cfrac{c_{1,2}}{1+\cfrac{c_{1,1}}{1+\cfrac{c_{1,2}}{1+\ddots}}}}+\cfrac{c_{2,1}}{1+\cfrac{c_{1,1}}{1+\cfrac{c_{1,2}}{1+\cfrac{c_{1,1}}{1+\ddots}}}+\cfrac{c_{2,1}}{1+\cfrac{c_{1,1}}{1+\cfrac{c_{1,2}}{1+\ddots}}+\cfrac{c_{2,1}}{1+\ddots+\ddots}}}$$
$$+\ldots+\cfrac{c_{N,1}}{1+\cfrac{c_{1,1}}{1+\cfrac{c_{1,2}}{1+\ddots}}+\cfrac{c_{2,1}}{1+\cfrac{c_{1,1}}{1+\ddots}+\cfrac{c_{2,1}}{1+\ddots}}+\ldots+\cfrac{c_{N,1}}{1+\cfrac{c_{1,1}}{1+\ddots}+\cfrac{c_{2,1}}{1+\ddots}+\ldots+\cfrac{c_{N,1}}{1+\ddots}}}. \tag{9}$$

For this we define tails of $\overrightarrow{p}$-periodic BCF (8) with initial conditions: $R_0^{(q,j)}=1$, $q=\overline{1,N}$, $1\le j\le n$, $n\ge 1$, and the recurrence relations

$$\begin{cases} R_n^{(1,s)}=1+\dfrac{c_{1,s}}{R_{n-1}^{(1,s+1)}}, & 1\le s\le p_1,\\ R_n^{(q,s)}=1+\sum_{k=1}^{q-1}\dfrac{c_{k,1}}{R_{n-1}^{(k,2)}}+\dfrac{c_{q,s}}{R_{n-1}^{(q,s+1)}}, & q=\overline{1,N}; 1\le s\le p_q, \end{cases} \tag{10}$$

where $n \geq 1$ $p_q \in \mathbb{N}$, $q = \overline{1,N}$. Then $R_n^{(q,j)} = R_n^{(q,s)}$ and $R_n^{(q,m)} = R_n^{(q-1,1)} + c_{q,m}/R_{n-1}^{(q,m+1)}$, $n \geq 1$, $q = \overline{1,N}$ $1 \leq j \leq n$, $1 \leq m \leq p_q - 1$, $p_q \in \mathbb{N}$.

Thus, the n-th approximants of BCF (8) are equal to $F_n = R_n^{(N,1)}$, $n \geq 1$, $F_0 = 1$.

For investigation of truncation error bounds of the fraction (8) we have used a formula for $n \geq 0$, $m > 0$, that had been proved in [1], such as

$$F_{n+m} - F_n = \sum_{\overrightarrow{k} \in \mathcal{I}_{n+1}^{(N)}} \frac{c_{1,1}^{k_{1,1}} c_{1,2}^{k_{1,2}} c_{2,1}^{k_{2,1}} \ldots c_{N,1}^{k_{N,1}}}{\prod_{j=1}^{k_1} \left(R_{m+l_1-j}^{(1,j+1)} \cdot R_{l_1-j}^{(1,j+1)} \right) \ldots \prod_{j=1}^{k_N} \left(R_{m+n-j}^{(N,1)} \cdot R_{n-j}^{(N,1)} \right)}, \tag{11}$$

where $\mathcal{I}_{n+1}^{(N)} := \left\{ \overrightarrow{k} = (k_1, k_2, \ldots, k_N) : k_1 = k_{1,1} + k_{1,2};\ k_l \geq 0;\ l = \overline{1,N};\ \sum_{l=1}^{N} k_l = n+1 \right\}$, $l_i = n - \sum_{t=i+1}^{N} k_t$, $R_{-1}^{(q,j)} = 1$, $q = \overline{1,N}$, $j = \overline{1,p_q}$, $p_1 = 2$, $p_2 = \ldots = p_N = 1$, $k_{r,s}$ is defined in [3]. Now we consider the 2-periodic continued fraction

$$1 + \frac{a|}{|1} + \frac{b|}{|1} + \frac{a|}{|1} + \frac{b|}{|1} + \ldots. \tag{12}$$

Let $\lambda = 1 + a + b$ and $\lambda \neq 0$. According to [5, Theorems 2.19, 2.20], [6, Theorem 1.6] we have that the even part and the odd part of the fraction (12) are equal to

$$1 + a \left(1 + b + \mathop{\mathrm{D}}_{k=2}^{\infty} \frac{c_k}{d_k} \right)^{-1}, \qquad 1 + a + \mathop{\mathrm{D}}_{k=1}^{\infty} \frac{c_k}{d_k}$$

respectively, where $c_k = -ab$, $d_k = 1 + a + b$. Next, let P_ν, Q_ν be the ν-th nominator and the ν-th denominator of 1-periodic continued fraction

$$1 + \frac{-ab/(1+a+b)^2|}{|1} + \frac{-ab/(1+a+b)^2|}{|1} + \ldots, \quad \nu \geq 1. \tag{13}$$

Then, according to formulas (5), we have for $k \geq 0$

$$P_k = \frac{\widetilde{x}^{k+2} - \widetilde{y}^{k+2}}{\widetilde{x} - \widetilde{y}}, \quad Q_k = \frac{\widetilde{x}^{k+1} - \widetilde{y}^{k+1}}{\widetilde{x} - \widetilde{y}},$$

where $\widetilde{x} = (1 + \sqrt{1 - 4ab/\lambda^2})/2$, $\widetilde{y} = (1 - \sqrt{1 - 4ab/\lambda^2})/2$.

Let $f_n^{(s)} = 1 + \mathop{\mathrm{D}}_{k=s}^{n+s-1} \frac{a_k}{1}$ be the s-th tail of the fraction (12), $n \geq 1$, $1 \leq s \leq n$, where $a_{2k-1} = a$, $a_{2k} = b$, $k \geq 1$. Then the following formulas

$$f_{2\nu}^{(j)} = \frac{\lambda P_{\nu-1}}{-a_j Q_{\nu-1} + \lambda P_{\nu-1}}, \quad \nu > 0, \quad f_{2\nu+1}^{(j)} = \frac{-a_{j+1} Q_\nu + \lambda P_\nu}{Q_\nu}, \quad \nu \geq 0, \quad j = 1, 2,$$

are valid for the 1-st and 2-nd tails of 2-periodic continued fraction (12).

Lemma. *Let the elements of 2-periodic fraction (12) satisfy the condition $-ab/\lambda^2 \in \Omega$, where $\lambda = 1 + a + b$, $\lambda \neq 0$, and Ω is defined by formula (2). Then:*

1. *the fraction (12) converges to value $x = (1 + a - b + \lambda\sqrt{1 - 4ab/\lambda^2})/2$;*

2. *if* $f_{2k+1}^{(j)} \neq 0$, $k \geq 0$, $j = \overline{1,2}$, *and* $|-a + \lambda P_k/Q_k| \geq \varepsilon_1 > 0$, $k \geq 0$, *then truncation error bounds are valid*

$$|f_n - x| \leq C q^{[(n+1)/2]}, \quad n \geq 0, \tag{14}$$

where $q = \left|\dfrac{1-\sqrt{1-4ab/\lambda^2}}{1+\sqrt{1-4ab/\lambda^2}}\right| < 1$, $C = \dfrac{|\widetilde{x}||\lambda|(1+q)^2}{(1-q)^2}\max\left\{\dfrac{1}{\varepsilon_2}; \dfrac{|a|}{\varepsilon_1^2}\right\}$ $\varepsilon_2 = |b| + |\lambda| M$, $M = |\widetilde{x}|(1+q^2)/(1-q)$, $\widetilde{x} = (1+\sqrt{1-4ab/\lambda^2})/2$.

Proof. Let $c = -ab/\lambda^2$. Since $c \in \Omega$, then 1-periodic continued fraction (13) converges and its value is $\widetilde{x}$, moreover $|\widetilde{x}| > |\widetilde{y}|$. Next, since $\lambda \neq 0$, then $\lim\limits_{\nu\to\infty} f_{2\nu+1} = \lim\limits_{\nu\to\infty} f_{2\nu} = x$. From this it follows that the fraction (12) converges and $\lim\limits_{n\to\infty} f_n = x$.

Since $c \in \Omega$, all approximants of the fraction (13) are not equal to zero. It follows that $f_{2n}^{(j)} \neq 0$, $n \geq 1$, $j = \overline{1,2}$. For $n \geq 1$ and $m \geq 1$ we estimate the difference $|f_{n+2m} - f_n|$, using formula (11). By virtue of $\left|\dfrac{P_k}{Q_k}\right| = \left|\dfrac{\widetilde{x}^{k+2} - \widetilde{y}^{k+2}}{\widetilde{x}^{k+1} - \widetilde{y}^{k+1}}\right| = |\widetilde{x}| \left|\dfrac{1-(\widetilde{y}/\widetilde{x})^{k+2}}{1-(\widetilde{y}/\widetilde{x})^{k+1}}\right|$ for $k \geq 0$ the following inequalities are valid $\mu \leq |P_k/Q_k| \leq M$, where $\mu = |\widetilde{x}|(1-q)$, and $|-b + \lambda P_k/Q_k| \leq \varepsilon_2$.

Let n and k be arbitrary natural numbers, moreover $n = 2r+1$, $k = r+m$, $r \geq 0$, $m \geq 0$. Then

$$|f_{2k+1} - f_{2r+1}| = \frac{|a|^{r+1}|b|^{r+1}}{\prod_{q=1}^{r+1}\left(|f_{2k-2q+2}^{(2)}||f_{2k-2q+1}^{(1)}||f_{2r+1-2q+1}^{(2)}||f_{2r+1-2q}^{(1)}|\right)},$$

$$\prod_{q=1}^{r+1}\left|f_{2(k-q+1)}^{(2)} f_{2(k-q)+1}^{(1)}\right| = |\lambda|^{r+1}\left|\frac{P_{k-1}}{-bQ_{k-r} + \lambda P_{k-r}}\right| \geq |\lambda|^{r+1}|\widetilde{x}|^r \frac{(1-q)}{(1+q)\varepsilon_2},$$

$$\prod_{q=1}^{r+1}\left|f_{2(r-q+1)}^{(2)} f_{2(r-q)+1}^{(1)}\right| = |\lambda|^r\,|P_{r-1}| \geq |\lambda|^r|\widetilde{x}|^r \frac{1-q}{1+q}.$$

From this, we have

$$|f_{2k+1} - f_{2r+1}| \leq \frac{(ab/\lambda^2)^{r+1}|\lambda|(1-q)^2}{|x|^{2r}(1+q)^2 M} = \frac{|\widetilde{y}|^{r+1}|\widetilde{x}|^{r+1}|\lambda|(1+q)^2}{|\widetilde{x}|^{2r+1}(1-q)^2 M} = C_1\left|\frac{\widetilde{y}}{\widetilde{x}}\right|^{r+1},$$

where $C_1 = |\lambda||x|(1+q)^2/((1-q)^2 M)$.

Let n and k be arbitrary natural numbers, moreover $n = 2r+1$, $k = r+m$, $r \geq 0$, $m \geq 0$. Then, by analogy we have

$$|f_{2k} - f_{2r}| \leq \frac{|a|^{r+1}|b|^r(1+q)^2}{|\lambda|^{2r}|\widetilde{x}|^{2r-1}(1-q)^2\varepsilon_1^2} = C_2\left|\frac{\widetilde{y}}{\widetilde{x}}\right|^r, \quad C_2 = \frac{|a||\widetilde{x}|(1+q)^2}{(1-q)^2\varepsilon_1^2}.$$

Finally, we obtain truncation error bounds (14) for $m \to \infty$. □

Now we consider the linear fractional transformation

$$t_1(\omega) = 1 + \cfrac{c_{1,1}}{1 + \cfrac{c_{1,2}}{\omega}}. \tag{15}$$

Let X_1 be the attracting fixed point of this transformation, X_j, Y_j be the attracting and repelling fixed points of $t_j(\omega) = X_{j-1} + c_{j,1}/\omega$, $j = \overline{2,N}$. It is known in [5, p. 190], that

$$X_1 = \left(\lambda - 2c_{1,2} + \lambda\sqrt{1 - 4c_{1,1}c_{1,2}/\lambda^2}\right)\Big/2. \tag{16}$$

Theorem. *Let $\mu = -c_{1,1}c_{1,2}/\lambda^2$, $\lambda = 1 + c_{1,1} + c_{1,2}$, $\lambda \neq 0$, $\mu \in \Omega_1$, where Ω_1 is defined by the formula (2), and let the elements of the fraction (9) satisfy the following conditions $c_{j,1} \in \Omega_j$, $j = \overline{2, N}$, where $\Omega_j = \{z \in \mathbb{C} : |\arg(z + X^2_{j-1}/4)| < \pi\}$. Then:*

1. *the fraction (9) converges and its value is $F = X_N$;*

2. *moreover, if $R^{(j,1)}_{2n+1} \neq 0$, $n \geq 0$, $j = 1, 2$; $| - c_{1,1} + \lambda P_k/Q_k| \geq \varepsilon_1 > 0$, $k \geq 1$,*

$$|c_{j,1}| < \frac{1}{4}\prod_{k=1}^{j-1} r_p, \quad j = \overline{2, N}, \tag{17}$$

where $r_1 = |\lambda||\widetilde{x}|\dfrac{(1-\rho_1)\,\varepsilon_1}{(1+\rho_1)\,\varepsilon_2}$, $r_k = \nu_k^2$, $\nu_k = (1 + d_k)/2$, $d_k = \sqrt{1 - 4|c_{k,1}|/\prod_m^{k-1} r_m}$, $k = \overline{2, N}$, $\widetilde{x} = (1 + \sqrt{1 - 4c_{1,1}c_{1,2}/\lambda^2})/2$, $\varepsilon_2 = |c_{1,2}| + |\lambda||\widetilde{x}|(1 + \rho_1^2)/(1 - \rho_1)$, then for $n \geq 1$ the truncation error bounds are valid

$$|F_n - F| \leq L \cdot \binom{N-1}{N+n-1} \cdot \frac{(\sqrt{\rho_1})^{n+1} - \rho^{n+1}}{\rho_1/\rho - \sqrt{\rho_1}}, \tag{18}$$

where $\rho_1 = \left|\dfrac{1 - \sqrt{1 - 4c_{1,1}c_{1,2}/\lambda^2}}{1 + \sqrt{1 - 4c_{1,1}c_{1,2}/\lambda^2}}\right|$, $\rho = \max\limits_{j=\overline{2,N}}\{\rho_j\}$, $\rho_j = \dfrac{1}{(1+d_j)^2}$, $L = \prod_{j=1}^N \dfrac{M_j}{\nu_j^4}$, $M_1 = \left(\dfrac{1+\rho_1}{1-\rho_1}\right)^2 \dfrac{\varepsilon_2}{\varepsilon_1\rho_1}$, $M_j = \max\left\{1, \dfrac{|c_{j,1}|}{\rho_j \prod_{m=1}^{j} \nu_m}\right\}$, $\nu_1 = \min\left\{\varepsilon_1, \dfrac{|\lambda||\widetilde{x}|(1-\rho_1)}{2\varepsilon_2}\right\}$, $j = \overline{2, N}$.

Proof. By induction on q we prove the convergence of the sequence $\{R^{(q,1)}_n\}_{n=1}^\infty$, $q = \overline{1, N}$.

For $q = 1$ the convergence of the sequence $\{R^{(1,1)}_n\}_{n=1}^\infty$ follows from Lemma, i.e. $\lim\limits_{n\to\infty} R^{(1,1)}_n = X_1$, where X_1 is defined by the formula (16). By induction hypothesis the following relations $\lim_{n\to\infty} R^{(k,1)}_n = X_k$, $X_k \neq 0$, $Y_k \neq 0$, hold for $q = k$, where $2 \leq k \leq N - 1$. We write $R^{(q,1)}_n$ for $q = k + 1$ and for the arbitrary natural n as follows

$$R^{(k+1,1)}_n = R^{(k,1)}_n + \frac{c_{k+1,1}|}{|R^{(k,1)}_{n-1}} + \ldots + \frac{c_{k+1,1}|}{|R^{(k,1)}_0}.$$

Since $c_{k+1,1} \in \Omega_{k+1}$, the linear fractional transformation $\widehat{t}_{k+1}(\omega) = \dfrac{c_{k+1,1}/X_k^2}{1+\omega}$ is loxodromic and from (C) of the [5, Theorem 4.13] we have, that $\lim\limits_{n\to\infty} R^{(k+1,1)}_n = X_{k+1}$, where $X_{k+1} = -y_{k+1}$ and y_{k+1} is the repelling fixed point of $\widehat{t}_{k+1}(\omega)$. Next, since $c_{k+1,1} \neq 0$, then $X_{k+1} \neq 0$, $Y_{k+1} \neq 0$. Hence, $\lim\limits_{n\to\infty} F_n = X_N$.

Let k and m be arbitrary integer numbers and $1 \leq k \leq m$, $m \geq 1$, $k = [k_1/2]$, where k_1 is defined by the formula (11). By virtue of $\lambda \neq 0$, $R^{(1,2)}_n = f^{(2)}_n$ and $R^{(1,1)}_n = f^{(1)}_n$, $n \geq 1$, we have

$$\prod_{j=1}^{k}\left|R^{(1,2)}_{2\nu+1}\right|\left|R^{(1,1)}_{2\nu}\right| \geq |\lambda|^k \left|\frac{-c_{1,1}Q_\nu + \lambda P_\nu}{-c_{1,1}Q_{\nu-k} + \lambda P_{\nu-k}}\right| \geq |\lambda|^k|\widetilde{x}|^k \frac{(1-\rho_1)\varepsilon_1}{(1+\rho_1)\varepsilon_2},$$

where $\nu = (m + l_1)/2 - j$ and l_1 is defined by formula (11). If $\nu = (m + 1 - l_1)/2 - j$, then

$$\prod_{j=1}^{k}\left|R^{(1,2)}_{2\nu+1}\right|\left|R^{(1,1)}_{2\nu}\right| \geq |\lambda|^k \left|\frac{-c_{1,1}Q_\nu + \lambda P_\nu}{-c_{1,1}Q_{\nu-k} + \lambda P_{\nu-k}}\right| \geq |\lambda|^k|\widetilde{x}|^k.$$

Next, we have

$$\prod_{j=1}^{k}\frac{|c_{1,1}c_{1,2}|}{|R^{(1,j+1)}_{2m+l_1-2j+1}R^{(1,j+1)}_{2m+l_1-2j}||R^{(1,j+1)}_{l_1-2j+1}R^{(1,j+1)}_{l_1-2j}|}\le\frac{1}{C^2}\prod_{j=1}^{k}\frac{|c_{1,1}c_{1,2}|/\lambda^2}{|\widetilde{x}|^2}=M_1\left|\frac{\widetilde{y}}{\widetilde{x}}\right|^k.$$

Moreover, according to Lemma the inequality $|R^{(1)}_n|\ge\nu_1$ holds.

Let n be arbitrary natural number. By induction on q we prove that the following inequalities are valid

$$|R^{(q,1)}_n|\ge\prod_{j=1}^{q}\nu_j,\quad q=\overline{2,N}. \tag{19}$$

For $q=2$ we can write the tail $R^{(2,1)}_n$ in the form

$$R^{(2,1)}_n=R^{(1,1)}_n+\frac{c_{2,1}|}{|R^{(1,1)}_{n-1}}+\ldots+\frac{c_{2,1}|}{|R^{(1,1)}_{0}}=R^{(1,1)}_n h^{(2,1)}_n,\quad n\ge1,$$

where for $r=2$ and

$$h^{(r,1)}_n=1+\frac{c_{r,1}/R^{(r-1,1)}_nR^{(r-1,1)}_{n-1}|}{|1}+\frac{c_{r,1}/R^{(r-1,1)}_{n-1}R^{(r-1,1)}_{n-2}|}{|1}+\ldots+\frac{c_{r,1}/R^{(r-1,1)}_{1}R^{(r-1,1)}_{0}|}{|1}. \tag{20}$$

From [2, Lemma 2] it follows: if elements of the reversed fractions $h^{(2,1)}_n$, $n\ge1$, satisfy the condition $|a_n|<|a|<1/4$, then the inequality $|h^{(2,1)}_n|\ge\nu_2$ holds. From this we have $\left|\frac{c_{2,1}}{R^{(1,1)}_nR^{(1,1)}_{n-1}}\right|<\frac{|c_{2,1}|}{r_1}<\frac{1}{4}$. Thus the inequality $|h^{(2,1)}_n|\ge\nu_2$ is valid. Moreover, $|R^{(2,1)}_n|\ge\nu_1\nu_2$.

By induction hypothesis the inequalities (19) hold for $q=s$, where $3\le s\le N-1$. We write $R^{(q,1)}_n$ for $q=s+1$ as follows

$$R^{(s+1,1)}_n=R^{(s,1)}_n+\frac{c_{s+1,1}|}{|R^{(s,1)}_{n-1}}+\ldots+\frac{c_{s+1,1}|}{|R^{(s+1,1)}_{0}}=R^{(s,1)}_n h^{(s+1,1)}_n,\quad n\ge1,$$

where $h^{(s+1,1)}_n$ is reversed continued fraction, that is defined by the formula (20). Its elements satisfy the conditions $\left|\frac{c_{s+1,1}}{R^{(s,1)}_nR^{(s,1)}_{n-1}}\right|<\frac{|c_{s+1,1}|}{\prod_{j=1}^{q}r_j}<\frac{1}{4}$, $r_j=\nu_j^2$. Thus, we have $|h^{(s+1,1)}_n|\ge\nu_{s+1}$, moreover, the following relations hold

$$\left|R^{(s+1,1)}_n\right|=\left|R^{(s,1)}_n\right||h^{(s+1,1)}_n|>\prod_{j=1}^{s+1}\nu_j.$$

To prove the inequality (18) we have to estimate the following relations

$$\prod_{r=1}^{k_j}\frac{|c_{j,1}|}{|R^{(j,1)}_{l_j+2m-r}R^{(j,1)}_{l_j-r}|}=M_j\prod_{r=1}^{[k_j/2]}\frac{|c_{j,1}|}{|R^{(j,1)}_{l_j+2m-2r+1}R^{(j,1)}_{l_j+2m-2r}|}\cdot\prod_{r=1}^{[k_j/2]}\frac{|c_{j,1}|}{|R^{(j,1)}_{l_j-2r+1}R^{(j,1)}_{l_j-2r}|},\quad j=\overline{2,N},$$

where k_j is defined by the formula (11). Since for the arbitrary natural n

$$\frac{|c_{j,1}|}{|R_n^{(j,1)} R_{n-1}^{(j,1)}|} = \frac{|c_{j,1}|/|R_n^{(j-1,1)} R_{n-1}^{(j-1,1)}|}{|h_n^{(j,1)} h_{n-1}^{(j,1)}|} < \frac{1/4}{\nu_j^2} = \frac{1}{(1+d_j)^2} = \rho_j, \quad j = \overline{2, N},$$

then for $n \geq 1$ and $m \geq 1$

$$|F_{n+2m} - F_n| \leq \sum_{k_1=0}^{n+1} \rho_1^{[k_1/2]} \rho^{n+1-k_1} \leq L \cdot \binom{N-1}{N+n-1} \cdot \frac{(\sqrt{\rho_1})^{n+1} - \rho^{n+1}}{\rho_1/\rho - \sqrt{\rho_1}}.$$

Finally, we obtain the truncation error bounds (18) for $m \to \infty$. □

REFERENCES

[1] Bodnar D.I., Bubniak M.M. *Multidimensional Generalization of Oval Theorem for Periodic Branched Continued Fractions of the Special Form.* Zbirn. prats Inst. matem. NANU, Matem. probl. mekhan. ta obchysl. mat. 2014, **11** (4), 54–67. (in Ukrainian)

[2] Bodnar D.I., Bubniak M.M. *Estimates of the rate of pointwise and uniform convergence for one-periodic branched continued fractions of a special form.* J. Math. Sci. 2015, **208** (3), 289–300. doi:10.1007/s10958-015-2446-x

[3] Bubniak M.M. *Truncation error bounds for the 1-periodic branched continued fraction of the special.* Carpatian Math. Publ. 2013, **5**(2), 187–195. doi:10.15330/cmp.5.2.187-195. (in Ukrainian)

[4] Dmytryshyn R.I. *Some properties of branched continued fractions of special form.* Carpathian Math. Publ. 2015, **7** (1), 72–77. doi:10.15330/cmp.7.1.72-77

[5] Lorentzen L., Waadeland H. Convergence theory. In: Continued fractions, 1. Atlantis Press, World Scientific, Amsterdam-Paris, 2008.

[6] Perron O. Die Lehre von den Kettenbrüchen. B. G. Teubner Verlag, Stuttgard, 1957.

[7] Wall H.S. Analytic theory of continued fractions. American Math. Soc., 2000.

Received 05.09.2015

Revised 18.11.2015

Боднар Д.І., Бубняк М.М. *Про збіжність* $(2, 1, \ldots, 1)$*-періодичного гіллястого ланцюгового дробу спеціального вигляду* // Карпатські матем. публ. — 2015. — Т.7, №2. — С. 148–154.

Означено $(2, 1, \ldots, 1)$-періодичний гіллястий ланцюговий дріб спеціального вигляду. Доведено ознаки збіжності 2-періодичного неперервного дробу та $(2, 1, \ldots, 1)$-періодичного дробу гіллястого ланцюгового дробу спеціального вигляду. Встановлено оцінку швидкості збіжності цього дробу при додаткових умовах.

Ключові слова і фрази: періодичні гіллясті ланцюгові дроби спеціального вигляду, збіжність.